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Le bricoleur, peut-on le faire ?
Le bricoleur, oui on peut !

Bob le Bricoleur et ses amis

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[1] La première page de mon introduction pourrait aussi vous intéresser...

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[2] *Voyage au coeur de l'espace-temps*, éditions First, quinze euros bien investis !
[3] Et puis, comment remercier les rapporteurs avant de leur avoir envoyé le manuscrit ?

Contents

Chapter 1

Introduction

L'objet de cette thèse est la construction de solutions dites haute-fréquences aux équations d'Einstein dans le vide, équations au coeur de la théorie de la relativité générale. Ce chapitre introductif contient une présentation mathématique des fondements géométriques et analytiques de cette théorie se resserrant progressivement autour de la question des métriques haute-fréquences et des contributions nouvelles de l'auteur à ce sujet. Avant de se lancer, prenons quelques lignes pour raconter brièvement l'origine de la théorie de la relativité générale.

Comme on l'a dit, la relativité générale est une théorie physique élaborée par Albert Einstein entre 1907 et 1915. Elle a pour but de décrire la gravitation. Avant Einstein, cette dernière était décrite par la théorie de la gravitation universelle d'Isaac Newton. Énoncée en 1687 dans l'ouvrage *Principia Mathematica*, cette théorie décrit la gravitation comme une interaction, une force, qui s'exerce entre les objets présents dans l'univers. Elle décrit de manière unifiée de nombreux phénomènes *a priori* distincts comme la chute des corps sur Terre ou le mouvement des astres sur la voûte céleste. La théorie de Newton est ainsi extrêmement puissante, et culmine au milieu du 19ème siècle à travers la découverte de la planète Neptune. La géante bleue n'est alors pas connue, et son existence est postulée pour expliquer la trajectoire perturbée d'Uranus. En supposant qu'Uranus subit l'influence gravitationnelle d'un corps inconnu décrite par les lois de Newton, Urbain Le Verrier prédit par le calcul la position de l'intrus. Les télescopes confirment les prédictions de l'astronome français, et Neptune est la première planète à être découverte par une formule mathématique !

Au tournant du 20ème siècle et après 200 ans de succès, la théorie de Newton se heurte à deux écueils. Le premier est observationnel : les astronomes de l'époque observent une légère rotation du périhélie de Mercure. Ce mouvement n'est pas prédit par la théorie de la gravitation universelle, malgré les tentatives des physiciens pour inclure l'influence des autres planètes sur l'orbite de Mercure. Le deuxième écueil se prénomme Albert Einstein. Le jeune physicien travaille alors à l'Office des Brevets de Berne et publie en 1905 une série de quatre articles qui révolutionnent sa discipline. Le troisième article présente la relativité restreinte, qui a pour but d'adapter la mécanique des corps à l'électromagnétisme de Maxwell. Cette nouvelle théorie a pour conséquence la finitude de la vitesse de propagation de toute interaction. Cela inclut aussi l'interaction gravitationnelle, ce qui entre directement en contradiction avec la théorie de Newton, qui prédit l'instantanéité des actions réciproques.

Pour ces deux raisons, Einstein se donne en 1907 pour projet de révolutionner notre compréhension de la gravité, en introduisant une forte dose de mathématique et de géométrie, poursuivant ainsi son voyage relativiste entamé en 1905. Le résultat de ses recherches est publié en novembre 1915, c'est la relativité générale telle que nous la connaissons encore aujourd'hui.

Le plan de cette introduction est le suivant. On introduit les outils géométriques nécessaires dans la Section 1.1. La relativité générale et les équations d'Einstein sont présentées dans la Section 1.2, où l'on discute aussi le problème de Cauchy. Les Sections 1.3 et 1.4 traitent des sujets plus spécifiques que sont les ondes gravitationnelles et la conjecture de Burnett. La Section 1.5 montre comment les approximations haute-fréquences sont utilisées dans d'autres domaines de l'analyse des équations aux dérivées partielles. Finalement, la Section 1.6 présente les trois résultats de cette thèse.

1.1 Géométrie lorentzienne

Nous commençons par décrire les fondements géométriques de la théorie de la relativité générale, c'est-à-dire la géométrie lorentzienne. On se base sur l'ouvrage de référence [O'N83]. On suppose connues les notions de variétés différentielles et d'espace tangent, et dans toute la suite, toutes les variétés et objets associés sont de classe C^∞.

1.1.1 Tenseurs et métriques

Les objets les plus importants de la relativité générale sont les tenseurs, et parmi eux se démarque le tenseur métrique d'une variété.

1.1.1.1 Définitions

Si $\mathcal{M}$ est une variété différentielle de dimension n, on note $T_p\mathcal{M}$ l'espace tangent à $\mathcal{M}$ en un point $p \in \mathcal{M}$ et $T_p\mathcal{M}^* = (T_p\mathcal{M})^*$ l'espace tangent dual. Définissons le fibré tangent et le fibré cotangent :

Définition 1.1.1. *Le fibré tangent $T\mathcal{M}$ est l'ensemble des vecteurs tangents à $\mathcal{M}$:*

$$T\mathcal{M} = \bigcup_{p\in\mathcal{M}} T_p\mathcal{M}.$$

On note $\pi : T\mathcal{M} \longrightarrow \mathcal{M}$ la projection qui à $T_p\mathcal{M}$ associe p. Le fibré cotangent $T\mathcal{M}^*$ est l'ensemble des covecteurs tangents à $\mathcal{M}$:

$$T\mathcal{M}^* = \bigcup_{p\in\mathcal{M}} T_p\mathcal{M}^*.$$

On note $\pi^* : T\mathcal{M}^* \longrightarrow \mathcal{M}$ la projection qui à $T_p\mathcal{M}^*$ associe p.

Les fibrés $T\mathcal{M}$ et $T\mathcal{M}^*$ sont des variétés différentielles de dimension $2n$. On peut maintenant définir ce qu'est un champs de vecteurs et une 1-forme :

Définition 1.1.2. *Un champs de vecteurs X est une application de $\mathcal{M}$ dans $T\mathcal{M}$ telle que $\pi \circ X = \mathrm{Id}_\mathcal{M}$. Une 1-forme est une application ω de $\mathcal{M}$ dans $T\mathcal{M}^*$ telle que $\pi^* \circ \omega = \mathrm{Id}_\mathcal{M}$.*

On note $\Gamma(\mathcal{M})$ l'ensemble des champs de vecteurs de $\mathcal{M}$ et $\Lambda^1(\mathcal{M})$ celui des 1-formes. Concrètement, un champs de vecteur est une application qui à chaque point $p \in \mathcal{M}$ associe un vecteur tangent $X_p \in T_p\mathcal{M}$. On note $C^\infty(\mathcal{M})$ l'ensemble des fonctions de classe C^∞ définies sur $\mathcal{M}$ et à valeurs dans $\mathbb{R}$. Nous définissons maintenant les tenseurs, objets centraux de la relativité générale :

Définition 1.1.3. *Un champs de tenseur T de type (r,s) sur $\mathcal{M}$ est une application*

$$T : \left(\Lambda^1(\mathcal{M})\right)^r \times (\Gamma(\mathcal{M}))^s \longrightarrow C^\infty(\mathcal{M})$$

qui est $C^\infty(\mathcal{M})$-multilinéaire. On note $\mathcal{T}^r_s(\mathcal{M})$ l'ensemble des champs de tenseur de type (r,s) sur $\mathcal{M}$. On parle aussi de tenseur r fois covariant et s fois contravariant, ou plus simplement de (r,s)-tenseur.

Un tenseur est une généralisation des notions précédentes au sens suivant : $\mathcal{T}^0_0(\mathcal{M})$ s'identifie à $C^\infty(\mathcal{M})$, $\mathcal{T}^1_0(\mathcal{M})$ s'identifie à $\Gamma(\mathcal{M})$ et $\mathcal{T}^0_1(\mathcal{M})$ s'identifie à $\Lambda^1(\mathcal{M})$. En utilisant la C^∞-multilinéarité et une fonction saut lisse, on peut montrer que les tenseurs sont des objets locaux, et en particulier on peut définir un tenseur en un point p.

Le tenseur le plus fondamental d'une variété est son tenseur métrique, plus simplement appelé la métrique de la variété.

Définition 1.1.4. *Une métrique lorentzienne g sur $\mathcal{M}$ est un $(0,2)$-tenseur tel que :*

- *g est symétrique : pour tout $X,Y \in \Gamma(\mathcal{M})$, $g(X,Y) = g(Y,X)$,*
- *pour tout $p \in \mathcal{M}$, g_p est une forme non-dégénérée sur $T_p\mathcal{M}$,*
- *pour tout $p \in \mathcal{M}$, g_p a $(-1,1,\dots,1)$ pour signature.*

Une définition similaire pourrait être donnée pour les variétés riemanniennes, qui sont des variétés munies d'une métrique riemannienne, c'est-à-dire des $(0,2)$-tenseurs symétriques non-dégénérés de signature $(1,\dots,1)$. Toutes les définitions qui vont suivre sont données dans le cadre lorentzien mais elles s'appliquent aussi dans le cas riemannien. Une spécificité du cas lorentzien est que le signe de $g(X,X)$ est *a priori* quelconque: si X est un champ de vecteur on dit que

- X est de type temps si $g(X,X) < 0$,
- X est de type espace si $g(X,X) > 0$,
- X est nul (ou de type lumière) si $g(X,X) = 0$.

Contrairement au cas riemannien, un champ de vecteur nul au sens précédent n'est pas nul au sens usuel $X = 0$.

1.1.1.2 Opérations sur les tenseurs

Comme on le voit, les tenseurs sont des objets intrinsèques à la variété. En particulier, on les a définis sans utiliser de système de coordonnées. Cependant, il s'avère que la manipulation des tenseurs dans un système de coordonnées est très pratique. On se donne un système de coordonnées $(x^\alpha) = (x^\alpha)_{\alpha=1,\dots,n}$ défini sur un ouvert $\mathcal{U}$ de $\mathcal{M}$, on peut définir sur $\mathcal{U}$ les champs de vecteurs $\partial_\alpha = \frac{\partial}{\partial x^\alpha}$ et les 1-formes $\mathrm{d}x^\alpha$ qui forment une base de $T_p\mathcal{M}$ et $T_p\mathcal{M}^*$ respectivement, pour tout $p \in \mathcal{U}$. On a la relation $\mathrm{d}x^\alpha(\partial_\beta) = \delta^\alpha_\beta$. Si T est un tenseur de type (r,s), il s'exprime localement comme :

$$T = T^{\alpha_1,\dots,\alpha_r}_{\beta_1,\dots,\beta_s} \partial_{\alpha_1} \otimes \cdots \otimes \partial_{\alpha_r} \otimes \mathrm{d}x^{\beta_1} \otimes \cdots \otimes \mathrm{d}x^{\beta_s}, \tag{1.1.1}$$

où on utilise la notation fondamentale

$$T^{\alpha_1,\dots,\alpha_r}_{\beta_1,\dots,\beta_s} = T\left(\mathrm{d}x^{\alpha_1},\dots,\mathrm{d}x^{\alpha_r},\partial_{\beta_1},\dots,\partial_{\beta_s}\right)$$

ainsi que la convention de sommation implicite d'Einstein si le même indice est en haut et en bas dans une formule. Les fonctions scalaires $T^{\alpha_1,\dots,\alpha_r}_{\beta_1,\dots,\beta_s}$ sont les composantes de T dans le système de coordonnées $(x^\alpha)_\alpha$. En particulier, une métrique s'exprime localement sous la forme $g = g_{\alpha\beta}\mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta$, que l'on abrègera en $g = g_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta$.

Comme g est non-dégénérée, on peut lui associer un tenseur de type $(2,0)$ de coordonnées $g^{\alpha\beta}$ vérifiant la relation $g^{\alpha\beta}g_{\beta\gamma} = \delta^\alpha_\gamma$. Ce tenseur est noté g^{-1}. Ainsi, $g_{\alpha\beta}$ et $g^{\alpha\beta}$ permettent de "monter" ou de "descendre" les indices, c'est-à-dire de changer le type d'un tenseur sans changer son nombre d'argument ($r+s$ pour un (r,s)-tenseur). Par exemple, si T est un $(2,0)$-tenseur, on peut lui associer :

- un $(1,1)$-tenseur, dont les composantes en coordonnées sont $T^\alpha{}_\beta = g_{\beta\gamma}T^{\alpha\gamma}$,
- un $(0,2)$-tenseur, dont les composantes en coordonnées sont $T_{\alpha\beta} = g_{\alpha\gamma}g_{\beta\delta}T^{\gamma\delta}$.

Une opération importante sur les tenseurs est la contraction par rapport à une paire d'indices où un des indices est covariant et l'autre est contravariant. Elle permet de passer d'un (r,s)-tenseur à un $(r-1,s-1)$-tenseur. Par exemple, si T est un (r,s)-tenseur, on définit $C^1_1(T)$ comme le $(r-1,s-1)$-tenseur défini par

$$C^1_1(T) = T^{k,\alpha_2,\dots,\alpha_r}_{k,\beta_2,\dots,\beta_s}\partial_{\alpha_2}\otimes\cdots\otimes\partial_{\alpha_r}\otimes\mathrm{d}x^{\beta_2}\otimes\cdots\otimes\mathrm{d}x^{\beta_s}.$$

Cette opération s'effectue directement sur les composantes du tenseur dans une base fixée mais est indépendante du choix de la base (ce qui ne serait pas le cas si on contractait deux indices covariants, ou deux indices contravariants).

Les équations de la relativité générale sont des équations différentielles faisant intervenir des tenseurs, expliquons donc comment on dérive un tenseur.

Définition 1.1.5. *Une dérivation de tenseurs est une famille d'applications*

$$D^r_s : \mathcal{T}^r_s(\mathcal{M}) \longrightarrow \mathcal{T}^r_s(\mathcal{M})$$

telle que :

- *pour tout* $(A,B) \in \mathcal{T}^{r_1}_{s_1}(\mathcal{M}) \times \mathcal{T}^{r_2}_{s_2}(\mathcal{M})$, $D^{r_1+r_2}_{s_1+s_2}(A\otimes B) = A\otimes D^{r_2}_{s_2}(B) + D^{r_1}_{s_1}(A)\otimes B$,
- *pour toute contraction* C^k_ℓ *et* $A \in \mathcal{T}^r_s(\mathcal{M})$, *on a* $D^{r-1}_{s-1}\left(C^k_\ell(A)\right) = C^k_\ell\left(D^r_s(A)\right)$.

Le lemme suivant explique comment définir une dérivation de tenseurs canonique si on sait comment dériver les fonctions et les champs de vecteurs.

Lemme 1.1.1. *Soit* $\tilde{D}^0_0$ *une dérivation des fonctions et* $\tilde{D}^1_0$ *une dérivation des champs de vecteurs. il existe une unique dérivation de tenseurs* D *telle que* $D^0_0 = \tilde{D}^0_0$ *et* $D^1_0 = \tilde{D}^1_0$ *et de plus, si* A *est un* (r,s)*-tenseur, on a :*

$$\begin{aligned}D^r_s(A)(\omega^1,\dots,\omega^r,X_1,\dots,X_s) = {} & D^0_0\left(A(\omega^1,\dots,\omega^r,X_1,\dots,X_s)\right) \\ & -\sum_{i=1}^r A\left(\omega^1,\dots,D^0_1(\omega^i),\dots,\omega^r r,X_1,\dots,X_s\right) \\ & -\sum_{i=1}^s A\left(\omega^1,\dots,\omega^r,X_1,\dots,D^1_0(X_i),\dots,X_s\right).\end{aligned} \tag{1.1.2}$$

Dans la section suivante, nous définissons deux notions importantes de dérivation des champs de vecteurs.

1.1.2 Connexions et dérivées de Lie

Définition 1.1.6. *Une connexion D sur $\mathcal{M}$ est une application de la forme*

$$\begin{aligned} D : \Gamma(\mathcal{M}) \times \Gamma(\mathcal{M}) &\longrightarrow \Gamma(\mathcal{M}) \\ (X, Y) &\longmapsto D_X Y \end{aligned}$$

telle que $X \mapsto D_X Y$ est $C^\infty(\mathcal{M})$-linéaire, $Y \mapsto D_X Y$ est $\mathbb{R}$-linéaire et pour toute fonction $f \in C^\infty(\mathcal{M})$, on a

$$D_X(fY) = f D_X Y + X(f)Y.$$

Sur une même variété $\mathcal{M}$, il y a plusieurs connexions, et l'objectif du théorème suivant et d'en choisir une de façon canonique, en lien avec la métrique :

Théorème 1.1.1. *Soit $(\mathcal{M}, g)$ une variété lorentzienne. Il existe une unique connexion D sur $\mathcal{M}$ telle que :*

- *D est sans torsion, i.e $[X, Y] = D_X Y - D_Y X$,*
- *D est compatible avec g, i.e $Xg(Y, Z) = g(D_X Y, Z) + g(Y, D_X Z)$.*

Cette connexion est appelée la connexion de Levi-Civita associée à g et est caractérisée par la formule de Koszul

$$\begin{aligned} 2g(D_V W, X) = {} & Vg(W, X) + Wg(X, V) - Xg(V, W) \\ & - g(V, [W, X]) + g(W, [X, V]) + g(X, [V, W]). \end{aligned}$$

Dans la suite, la notation D sera réservée à la connexion de Levi-Civita. Si X est un champs de vecteur et f une fonction scalaire, on définit $D_X f = Xf$. Grâce au Lemme 1.1.1, on peut alors dériver n'importe quel tenseur en utilisant la connexion de Levi-Civita. En particulier, on peut dériver le tenseur métrique lui-même, on obtient

$$D_X g(Y, Z) = Xg(Y, Z) - g(D_X Y, Z) - g(Y, D_X Z) = 0, \tag{1.1.3}$$

où on a utilisé (1.1.2) et la condition de compatibilité du Théorème 1.1.1.

On peut appliquer la connexion de Levi-Civita aux champs de vecteurs ∂_α associés à un système de coordonnées. Les symboles de Christoffel donnent la décomposition de $D_\alpha \partial_\beta := D_{\partial_\alpha} \partial_\beta$ dans la base $(\partial_\gamma)_\gamma$. Le lemme suivant donne leur expression.

Lemme 1.1.2. *Si on se place dans un système de coordonnées, on a $D_\alpha \partial_\beta = \Gamma^\gamma_{\alpha\beta} \partial_\gamma$ où*

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2} g^{\gamma\rho} \left(\partial_\alpha g_{\beta\rho} + \partial_\beta g_{\alpha\rho} - \partial_\rho g_{\alpha\beta} \right). \tag{1.1.4}$$

En particulier, on a $\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\beta\alpha}$.

Une autre notion de dérivation utile est la dérivée de Lie.

Définition 1.1.7. *Soit $\phi : U \longrightarrow V$ un difféomorphisme entre deux ouverts de $\mathcal{M}$. Soit $Y \in \Gamma(V)$, on définit $\phi^* Y \in \Gamma(U)$ par $\phi^* Y_p = (\mathrm{d}_p \phi)^{-1} \left(Y_{\phi(p)} \right)$ pour tout $p \in U$. La dérivée de Lie de Y dans la direction X est*

$$\mathcal{L}_X Y = \frac{\mathrm{d}}{\mathrm{d}t} \left(\phi^X_{-t} \right)^* Y_{|t=0},$$

où ϕ^X est le flot de X, c'est-à-dire la solution de l'équation différentielle $\frac{\mathrm{d}}{\mathrm{d}t} \phi^X_t(p) = X_{\phi^X_t(p)}$, avec $\phi^X_0 = \mathrm{Id}_\mathcal{M}$, qui est bien un difféomorphisme local partout où elle est définit.

On peut démontrer que $\mathcal{L}_X Y = [X, Y]$. Si on définit $\mathcal{L}_X f = Xf$, alors on peut utiliser le Lemme 1.1.1 pour définir $\mathcal{L}_X T$ pour n'importe quel tenseur. En particulier dans un système de coordonnées on a

$$\mathcal{L}_\alpha g_{\beta\gamma} = \partial_\alpha g_{\beta\gamma} - g\left([\partial_\alpha, \partial_\beta], \partial_\gamma\right) - g\left(\partial_\beta, [\partial_\alpha, \partial_\gamma]\right) = \partial_\alpha g_{\beta\gamma} \tag{1.1.5}$$

où on a utilisé (1.1.2) et la commutation des champs de vecteurs ∂_α.

1.1.3 Le tenseur de courbure

La commutation des champs de vecteurs ∂_α associés à un système de coordonnées, que l'on vient d'utiliser, est connue sous le nom de règle de Schwarz. La dérivée covariante, autre nom de la connexion de Levi-Civita définie au Théorème 1.1.1, ne vérifie pas cette propriété, c'est-à-dire qu'en général

$$D_X D_Y Z \neq D_Y D_X Z.$$

Pour mesurer ce défaut de commutativité, on définit le tenseur de courbure, ou tenseur de Riemann.

Définition 1.1.8. *Soit $(\mathcal{M}, g)$ une variété lorentzienne et D la connexion de Levi-Civita. La fonction*

$$\begin{aligned} R : \Gamma(\mathcal{M})^3 &\longrightarrow \Gamma(\mathcal{M}) \\ (X, Y, Z) &\longmapsto R(X, Y)Z := D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z, \end{aligned}$$

définit un $(1,3)$-tenseur appelé le tenseur de Riemann de $(\mathcal{M}, g)$.

C'est l'ajout du terme $-D_{[X,Y]}Z$ qui permet à R de définir un tenseur, c'est-à-dire d'être C^∞-multilinéaire. Ses composantes en coordonnées sont définies par la relation

$$R(\partial_\beta, \partial_\gamma)\partial_\delta = R^\alpha{}_{\delta\beta\gamma}\partial_\alpha.$$

Elles vérifient certaines symétries

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} = -R_{\alpha\beta\delta\gamma} \tag{1.1.6}$$

ainsi que la première identité de Bianchi

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0.$$

Elles vérifient de plus une identité différentielle, appelée la deuxième identité de Bianchi

$$D_\nu R_{\alpha\beta\gamma\delta} + D_\alpha R_{\beta\nu\gamma\delta} + D_\beta R_{\nu\alpha\gamma\delta} = 0. \tag{1.1.7}$$

Le tenseur de Riemann est un tenseur à quatre indices et est donc compliqué à manipuler. En contractant des indices, on peut obtenir un objet plus simple. Au vu des symétries (1.1.6), la seule contraction intéressante est celle entre le premier et le troisième indice, ce qui définit le tenseur de Ricci.

Définition 1.1.9. *Le tenseur de Ricci de $(\mathcal{M}, g)$ est un $(0,2)$-tenseur donné en coordonnées par*

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu},$$

et la courbure scalaire de $(\mathcal{M}, g)$ est la trace du tenseur de Ricci, i.e $R = g^{\mu\nu}R_{\mu\nu}$.

En utilisant les symboles de Christoffel, la définition du tenseur de Riemann et en contractant une fois, on obtient l'expression suivante pour le tenseur de Ricci :

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\mu \Gamma^\alpha_{\nu\alpha} + \Gamma^\alpha_{\alpha\beta}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha}. \tag{1.1.8}$$

1.2 Relativité générale

Dans la section précédente, nous avons mis en place l'attirail géométrique nécessaire à la théorie de la relativité, que nous présentons maintenant.

1.2.1 Les équations d'Einstein

L'idée centrale de cette théorie est une description géométrique de la gravitation, qui devient la manifestation de la courbure de l'espace-temps, c'est-à-dire d'une variété lorentzienne $(\mathcal{M}, g)$ de dimension 4. La courbure de l'espace-temps doit être reliée à son contenu en matière et énergie. De plus, le principe de covariance doit être vérifié. Ce dernier stipule que les lois de la physique ne doivent pas dépendre du système de coordonnées dans lesquelles on les écrit. Les tenseurs définis à la section précédente sont donc *a priori* les objets parfaits pour exprimer les lois de la relativité générale, en particulier le tenseur de courbure et ses différentes contractions. Dernière contrainte : les tenseurs considérés doivent être de divergence nulle, pour respecter la conservation de la masse et de l'énergie.

Lemme 1.2.1. *On définit le tenseur d'Einstein par*

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}.$$

On a $\mathrm{div}_g G = 0$.

Démonstration. Grâce à (1.1.3) on a

$$\mathrm{div}_g G_\alpha = g^{\mu\nu} D_\mu G_{\nu\alpha} = g^{\mu\nu} D_\mu R_{\nu\alpha} - \frac{1}{2}\partial_\alpha R. \tag{1.2.1}$$

Pour démontrer que cette expression s'annule, on contracte la deuxième identité de Bianchi (1.1.7) sur les indices α et γ pour obtenir

$$D_\nu R_{\beta\delta} + D^\gamma R_{\beta\nu\gamma\delta} - D_\beta R_{\nu\delta} = 0.$$

On contracte cette expression sur les indices β et δ pour obtenir

$$\partial_\nu R - D^\gamma R_{\nu\gamma} - D^\delta R_{\nu\delta} = 0$$

ce qui conclut la preuve en rappelant (1.2.1). □

Les **équations d'Einstein de la relativité générale** s'énoncent alors ainsi

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}. \tag{1.2.2}$$

Au membre de droite de cette équation apparaît le tenseur énergie-impulsion, qui modélise le contenu en matière et en énergie de l'espace-temps. C'est un 2-tenseur symétrique de divergence nulle. Donnons deux exemples:

- **Le vide.** Si l'espace-temps est vide, on a $T_{\mu\nu} = 0$. Prenons alors la trace de (1.2.2), on obtient $R - 2R = 0$ (car la trace de la métrique est égale à la dimension de l'espace-temps, ici 4) puis $R = 0$. Les équations d'Einstein dans le vide se réécrivent donc

  $$R_{\mu\nu} = 0. \tag{1.2.3}$$

  Ce sont ces équations que l'on étudie dans cette thèse. La solution la plus simple est l'espace-temps de Minkowski, c'est-à-dire $\mathbb{R}^{3+1}$ muni de la métrique $-(\mathrm{d}t)^2 + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2$. Il s'agit de l'espace-temps plat de la relativité restreinte, qui est donc un cas particulier de la relativité générale.

- **La théorie cinétique relativiste.** Si l'espace-temps est rempli d'un gaz de particules se déplaçant à la vitesse de la lumière et pouvant être décrit pas une densité f, on a alors

$$T_{\mu\nu}(f) = \int_{\{p \in T\mathcal{M} \ | \ g(p,p)=0\}} f p_\mu p_\nu, \tag{1.2.4}$$

où la densité f est définie sur l'espace des phases $\mathcal{M} \times T\mathcal{M}$. La condition de divergence nulle de $T_{\mu\nu}(f)$ est équivalente à l'équation de Liouville-Vlasov pour f

$$p^\alpha \partial_{x^\alpha} f - \Gamma^\alpha_{\mu\nu} p^\mu p^\nu \partial_{p^\alpha} f = 0. \tag{1.2.5}$$

Si $T_{\mu\nu}(f)$ est défini par (1.2.4), le système d'inconnues g et f constitué de $G_{\mu\nu} = T_{\mu\nu}(f)$ et de (1.2.5) est appelé système Einstein-Vlasov de masse nulle. Voir le Chapitre 10 de [CB09] pour plus de détails sur ce système.

On peut aussi imaginer un couplage entre les équations d'Einstein et les équations d'Euler de la mécanique des fluides (voir Chapitre 9 de [CB09]) ou les équation de Maxwell de l'électromagnétisme.

Nous faisons maintenant trois remarques formelle sur les équations (1.2.2).

1. Les équations d'Einstein (1.2.2) peuvent s'interpréter comme l'équation d'Euler-Lagrange pour la fonctionnelle (appelée action d'Einstein-Hilbert)

$$S_\mathcal{M}[g] = \int_\mathcal{M} \left(R[g] + L_m[\varphi, g] \right) \mathrm{d}\mu_g, \tag{1.2.6}$$

où $R[g]$ est la courbure scalaire associée à g, $L_m[\varphi, g]$ correspond correspond à l'action de la matière présente dans l'espace-temps et où $\mu_g = \sqrt{-\det(g)}\mathrm{d}x$ est la mesure invariante associée à g.

2. Une autre façon de justifier l'expression du tenseur d'Einstein est de chercher tous les 2-tenseurs $A_{\mu\nu}$ tels que :

   - A est symétrique et de divergence nulle,
   - A ne dépend que de g et de ses deux premières dérivées et est linéaire en $\partial^2 g$.

   Un théorème de Lovelock (voir [Lov72]) assure que les seuls tenseurs vérifiant ces quatre conditions sont les combinaisons linéaires de $G_{\mu\nu}$ et $g_{\mu\nu}$. Cela éclaire le choix d'Einstein d'ajouter le terme $\Lambda g_{\mu\nu}$ à (1.2.2), où Λ est la fameuse constante cosmologique.

3. Les équations (1.2.2) sont tensorielles et sont donc invariantes par difféomorphisme, c'est-à-dire vraies dans n'importe quel système de coordonnées. Pour les résoudre, on doit choisir un système de coordonnées privilégié, autrement dit faire un choix de jauge.

On aimerait comprendre la structure du système (1.2.3) afin de formuler un problème de Cauchy. La proposition suivante donne une expression du tenseur de Ricci faisant appel à l'opérateur d'onde, ou d'Alembertien, associé à g, qui est défini par

$$\Box_g f := g^{\mu\nu} D_\mu D_\nu f = g^{\mu\nu} \left(\partial_\mu \partial_\nu f - \Gamma_{\mu\ \nu}^{\ \alpha} \partial_\alpha f \right). \tag{1.2.7}$$

On définit aussi $\tilde{\Box}_g f := g^{\mu\nu} \partial_\mu \partial_\nu f$, c'est-à-dire la partie principale du d'Alembertien. Dans le cas de l'espace-temps de Minkowski, le d'Alembertien est l'opérateur d'onde classique $\Box =$

$-\partial_t^2 + \Delta$. On introduit aussi la notation de symétrisation suivante pour une expression quelconque dépendant de deux indices

$$T_{(\alpha\beta)} = T_{\alpha\beta} + T_{\beta\alpha}.$$

Proposition 1.2.1. *On a*

$$R_{\mu\nu} = -\frac{1}{2}\tilde{\Box}_g g_{\mu\nu} + \frac{1}{2} g_{\rho(\mu}\partial_{\nu)} H^\rho + Q_{\mu\nu}(\partial g, \partial g) \tag{1.2.8}$$

où $Q_{\mu\nu}(\partial g, \partial g)$ est une forme quadratique en les coefficients de la métrique et où

$$H^\rho = g^{\alpha\beta}\Gamma^\rho_{\alpha\beta}. \tag{1.2.9}$$

Démonstration. On part de (1.1.8). Notons que les produits $\Gamma\Gamma$ des symboles de Christoffel sont des termes quadratiques en les coefficients de la métrique (cf (1.1.4)), ils contribuent donc à la définition de $Q_{\mu\nu}$ et nous ne les développons pas. De même pour les termes $\partial\Gamma$, si la dérivée tombe sur l'inverse de g alors cela contribue à $Q_{\mu\nu}$. Les seuls termes de $R_{\mu\nu}$ qui ne sont pas de la forme $g^{-2}\partial g \partial g$ proviennent donc de $\partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\mu \Gamma^\alpha_{\nu\alpha}$ et en particulier du cas où les dérivées impactent ∂g. En utilisant (1.1.4), ces termes sont égaux à

$$\begin{aligned} &\frac{1}{2} g^{\alpha\beta}\left(\partial_\alpha\partial_\mu g_{\nu\beta} + \partial_\alpha\partial_\nu g_{\mu\beta} - \partial_\alpha\partial_\beta g_{\mu\nu}\right) - \frac{1}{2} g^{\alpha\beta}\left(\partial_\mu\partial_\nu g_{\alpha\beta} + \partial_\mu\partial_\alpha g_{\nu\beta} - \partial_\mu\partial_\beta g_{\nu\alpha}\right) \\ &\quad = -\frac{1}{2}\tilde{\Box}_g g_{\mu\nu} + \frac{1}{2} g^{\alpha\beta}\left(\partial_\alpha\partial_\mu g_{\nu\beta} + \partial_\alpha\partial_\nu g_{\mu\beta} - \partial_\mu\partial_\nu g_{\alpha\beta}\right). \end{aligned}$$

De plus si on s'intéresse uniquement à la partie quasi-linéaire (c'est-à-dire de la forme $g^{-1}\partial^2 g$) de $\frac{1}{2} g_{\rho(\mu}\partial_{\nu)} H^\rho$ on obtient

$$\frac{1}{4} g_{\rho(\mu} g^{\alpha\beta} g^{\rho\sigma}\left(\partial_{\nu)}\partial_\alpha g_{\beta\sigma} + \partial_{\nu)}\partial_\beta g_{\alpha\sigma} - \partial_{\nu)}\partial_\sigma g_{\alpha\beta}\right) = \frac{1}{2} g^{\alpha\beta}\left(\partial_\nu\partial_\alpha g_{\beta\mu} + \partial_\mu\partial_\alpha g_{\beta\nu} - \partial_\nu\partial_\mu g_{\alpha\beta}\right)$$

ce qui conclut la preuve. □

1.2.2 Le problème de Cauchy en relativité générale

Nous allons maintenant définir le problème de Cauchy en relativité générale, c'est-à-dire pour les équations (1.2.3) (une construction similaire pourrait être donnée dans le cas plus général de (1.2.2)). L'expression (1.2.8) montre que si $H^\rho = 0$, les équations (1.2.3) se réécrivent

$$\tilde{\Box}_g g_{\mu\nu} = 2Q_{\mu\nu}(\partial g, \partial g), \tag{1.2.10}$$

c'est-à-dire sous la forme d'un système d'équation d'ondes quasi-linéaires avec une non-linéarité quadratique en les coefficients de la métrique. On va donc s'inspirer du problème de Cauchy pour l'équation d'onde classique

$$\Box u = F(u, \partial u) \tag{1.2.11}$$

sur $\mathbb{R}_+ \times \mathbb{R}^3$. Pour cette équation, les données de Cauchy sont deux fonctions φ_0 et φ_1 définies sur $\mathbb{R}^3$, et résoudre le problème de Cauchy pour (1.2.11) consiste à trouver u définie sur $\mathbb{R}_+ \times \mathbb{R}^3$ vérifiant (1.2.11) et telle que

$$u \restriction \{t = 0\} = \varphi_0 \quad \text{et} \quad \partial_t u \restriction \{t = 0\} = \varphi_1.$$

Nous allons adapter cette construction au cadre géométrique des équations d'Einstein dans le vide. Nous devons définir l'analogue de l'hypersurface $\{t = 0\}$ et des restrictions $u \restriction \{t = 0\}$ et $\partial_t u \restriction \{t = 0\}$.

1.2.2.1 Hypersurface de type espace

Dans l'espace-temps de Minkowski $\mathbb{R}^{3+1}$, l'hypersurface $\{t=0\}$ est de type espace, c'est-à-dire que les vecteurs tangents à $\{t=0\}$ sont de type espace pour la métrique de Minkowski. Cette définition se généralise au cas d'une variété lorentzienne $(\mathcal{M},g)$ quelconque : une hypersurface $\Sigma\subset\mathcal{M}$ est dite de type espace si $g(X,X)>0$ pour tout $X\in\Gamma(\Sigma)$. C'est sur une hypersurface de type espace que nous allons définir les données de Cauchy pour (1.2.3).

On peut associer à Σ sa normale unitaire, c'est-à-dire l'unique champs de vecteur T de type temps vérifiant $g(T,T)=-1$ et $g(T,X)=0$ pour tout $X\in\Gamma(\Sigma)$. L'objet suivant est fondamental.

Définition 1.2.1. *Soit $\Sigma\subset\mathcal{M}$ une hypersurface de type espace et T la normale unitaire à Σ. On définit un 2-tenseur sur Σ par*

$$K(X,Y)=-g(D_XT,Y),$$

que l'on appelle la seconde forme fondamentale de Σ.

La proposition suivante nous permet d'interpréter la seconde forme fondamentale comme la dérivée de la métrique dans la direction orthogonale à Σ :

Proposition 1.2.2. *Le tenseur K est symétrique et vérifie*

$$K=-\frac{1}{2}\mathcal{L}_Tg. \tag{1.2.12}$$

Démonstration. On vérifie la symétrie de K : pour $X,Y\in\Gamma(\Sigma)$ on utilise la condition de compatibilité pour obtenir

$$\begin{aligned}K(X,Y)-K(Y,X)&=g(T,D_XY)-g(T,D_YX)\\&=g(T,[X,Y])\\&=0,\end{aligned}$$

où on a utilisé le fait que D est sans torsion et que $[X,Y]\in\Gamma(\Sigma)$. Pour (1.2.12), on utilise de plus (1.1.2):

$$\begin{aligned}\mathcal{L}_Tg(X,Y)&=Tg(X,Y)-g([T,X],Y)-g(X,[T,Y])\\&=g(D_TX,Y)+g(X,D_TY)-g(D_TX,Y)\\&\quad+g(D_XT,Y)-g(X,D_TY)+g(X,D_YT)\\&=-K(X,Y)-K(Y,X)\end{aligned}$$

ce qui conclut la preuve en utilisant la symétrie de K. □

On peut restreindre la métrique g à Σ car $T_p\Sigma$ est un sous-espace vectoriel de $T_p\mathcal{M}$ pour tout $p\in\Sigma$. On définit ainsi la métrique riemannienne $\bar{g}$. À cette métrique sont associés une dérivée covariante $\bar{D}$ et un tenseur de courbure $\bar{R}$ définis comme dans le cas lorentzien. On utilise les lettres latines $i,j,k,\ell\ldots$ pour noter un système de coordonnées quelconque sur Σ. La proposition suivante fait le lien entre le point de vue extrinsèque et intrinsèque sur Σ, c'est-à-dire calcule la courbure de Σ vu depuis $\mathcal{M}$ en fonction de $\bar{R}$ et K.

Proposition 1.2.3. *L'équation de Gauss est vérifiée:*

$$R_{ij\ell m} = \bar{R}_{ij\ell m} + K_{mj}K_{\ell i} - K_{\ell j}K_{mi}. \tag{1.2.13}$$

L'équation de Codazzi est vérifiée:

$$R_{i0j\ell} = -\bar{D}_j K_{i\ell} + \bar{D}_\ell K_{ij}. \tag{1.2.14}$$

On en déduit les composantes temporelles du tenseur d'Einstein Dans la base (T, e_i), c'est-à-dire G_{00} et G_{0i}.

Corollaire 1.2.1. *On a*

$$\begin{aligned} 2G_{00} &= \bar{R} + (\mathrm{tr}_{\bar{g}} K)^2 - |K|^2_{\bar{g}}, \\ G_{0i} &= -\bar{D}^\ell K_{i\ell} + \bar{D}_i \mathrm{tr}_{\bar{g}} K, \end{aligned}$$

où l'on a posé $|K|^2_{\bar{g}} = \bar{g}^{ij}\bar{g}^{k\ell}K_{ik}K_{j\ell}$.

1.2.2.2 Les données et le problème de Cauchy

Nous pouvons maintenant définir les données de Cauchy pour (1.2.3) par analogie à l'équation (1.6.6). L'analogue de $\{t = 0\}$ est une variété Σ de dimension 3 et l'analogue de $u \restriction \{t = 0\}$ est une métrique riemannienne $\bar{g}$ sur Σ. Pour la dérivée dans une direction orthogonale, on ne peut choisir la dérivée covariante car $Dg = 0$ (cf (1.1.3)), on choisit donc la dérivée de Lie et on prescrit $\mathcal{L}_T g$. Par la Proposition 1.2.2 cela revient à prescrire un 2-tenseur symétrique K sur Σ.

Le triplet $(\Sigma, \bar{g}, K)$ définit les données de Cauchy. Le problème de Cauchy est alors : chercher une variété lorentzienne $(\mathcal{M}, g)$ de tenseur de Ricci identiquement nul et telle que $\Sigma \subset \mathcal{M}$ soit de type espace, $\bar{g}$ soit la restriction de g à Σ et K soit la seconde forme fondamentale de Σ.

Comme le montre le Corollaire 1.2.1, les composantes temporelles G_{00} et G_{0i} ne dépendent que de $\bar{g}$ et K. Il est donc nécessaire que les données de Cauchy vérifient les équations suivantes sur Σ, dites équations de contraintes :

$$\bar{R} + (\mathrm{tr}_{\bar{g}} K)^2 - |K|^2_{\bar{g}} = 0, \tag{1.2.15}$$

$$-\bar{D}^\ell K_{i\ell} + \bar{D}_i \mathrm{tr}_{\bar{g}} K = 0. \tag{1.2.16}$$

L'équation (1.2.15) est appelée la contrainte hamiltonienne et (1.2.16) la contrainte de moment. Dans la section suivante, nous allons montrer que résoudre les équations de contraintes est aussi une condition suffisante pour construire une solution à (1.2.3).

Comme un des théorèmes de cette thèse concerne les équations de contraintes, nous revenons plus en détails sur ces dernières et sur une méthode de résolution particulière, la méthode conforme, dans la Section 1.2.3.

1.2.2.3 Existence locale en coordonnées d'ondes

Nous montrons maintenant comment construire une solution aux équations d'Einstein dans le vide en utilisant les coordonnées d'ondes (voir aussi le Chapitre 10 de [Wal84]). Ce choix de jauge correspond à choisir des coordonnées $(x^\alpha)_\alpha$ telles que $H^\rho = 0$ (cf (1.2.9)), ce qui est équivalent à avoir $\Box_g x^\alpha = 0$.

Pour simplifier la présentation, on suppose que l'hypersurface initiale est $\mathbb{R}^3$. On va construire une solution aux équations d'Einstein de la forme $([0,T]\times\mathbb{R}^3, g)$. On procède de la manière suivante:

1. On résout les équations de contraintes (1.2.15)-(1.2.16) et on obtient $(\bar{g}, K)$. À partir de $\bar{g}$ on définit $g_{\alpha\beta} \restriction \{t=0\}$:

   - pour les composantes spatiales on choisit $g_{ij} = \bar{g}_{ij}$,
   - pour les composantes temporelles on choisit $g_{00} = -1$ et $g_{0i} = 0$ de telle sorte que ∂_t est la normale unitaire.

   À partir de K on définit $\partial_t g_{ij} \restriction \{t=0\}$. Comme ∂_t est la normale unitaire, on utilise (1.2.12) et (1.1.5) et on pose $\partial_t g_{ij} = -2K_{ij}$. Cela assure que K est bien la seconde forme fondamentale de $\{t=0\}$. De plus on choisit $\partial_t g_{0\alpha} \restriction \{t=0\}$ de telle sorte que $H^\rho \restriction \{t=0\} = 0$. C'est possible car sur $\{t=0\}$ on a

$$\begin{aligned} 2H^0 &= \partial_t g_{00} - 2\mathrm{tr}_{\bar{g}} K, \\ \bar{g}_{ij} H^i &= -\partial_t g_{0j} + \bar{g}^{k\ell}\left(\partial_k \bar{g}_{\ell j} - \frac{1}{2}\partial_j \bar{g}_{k\ell}\right). \end{aligned}$$

2. On résout le système d'équations d'ondes suivant pour les coefficients $g_{\alpha\beta}$:

$$\tilde{\Box}_g g_{\mu\nu} = 2Q_{\mu\nu}(\partial g, \partial g), \tag{1.2.17}$$

   avec les choix précédemment faits de données initiales. On résout (1.2.17) grâce à un schéma itératif basé sur des estimations d'énergie pour l'opérateur $\tilde{\Box}_g$. On ne donne pas plus de détails. On obtient une métrique g sur $[0,T]\times\mathbb{R}^3$ pour un certain $T>0$.

3. Il reste à montrer que le tenseur de Ricci de g s'annule. Grâce à (1.2.17) et (1.2.8), g vérifie

$$R_{\mu\nu} = \frac{1}{2} g_{\rho(\mu}\partial_{\nu)} H^\rho$$

   ce qui implique

$$2G_{\mu\nu} = g_{\rho(\mu}\partial_{\nu)} H^\rho - \partial_\rho H^\rho g_{\mu\nu}. \tag{1.2.18}$$

   Or, puisqu'on a $H^\rho \restriction \{t=0\} = 0$, on obtient sur $\{t=0\}$

$$\begin{aligned} 2G_{00} &= -\partial_t H^0, \\ 2g^{ij} G_{0i} &= \partial_t H^j. \end{aligned}$$

   Les équations de contraintes étant vérifiées, on obtient $\partial_t H^\rho \restriction \{t=0\} = 0$. On utilise maintenant le fait que $\mathrm{div} G = 0$ (cf Lemme 1.2.1). Schématiquement, si on injecte (1.2.18) dans cette équation on obtient

$$\Box_g H^\rho + F^\rho(H, \partial H) = 0$$

   sur $[0,T]\times\mathbb{R}^3$. Comme H et $\partial_t H$ sont nulles initialement, on obtient $H^\rho = 0$ dans tout l'espace-temps et $G = 0$ en rappelant (1.2.18). On a bien résolu (1.2.3) en coordonnées d'ondes.

Résumons cette stratégie : pour résoudre (1.2.3) dans une jauge particulière on résout les contraintes et choisit des données initiales telles que les conditions de jauges soient initialement vérifiées, on résout un problème d'évolution qui correspond à annuler le tenseur de Ricci en négligeant les termes de jauges, puis on propage les conditions de jauges en utilisant div$G = 0$ ce qui finit d'annuler le tenseur de Ricci. C'est ainsi que Choquet-Bruhat démontra dans [FB52] la partie "existence" du théorème fondamental suivant :

Théorème 1.2.1. *Tout triplet $(\Sigma, \bar{g}, K)$ de données de Cauchy suffisamment lisses satisfaisant les équations de contraintes admet un unique développement $(\mathcal{M}, g)$ maximal globalement hyperbolique satisfaisant (1.2.3).*

La partie "unicité" du théorème précédent est due à Choquet-Bruhat et Geroch dans [CBG69].

1.2.3 Les équations de contraintes et la méthode conforme

Dans cette section, nous revenons sur les équations de contraintes (1.2.15)-(1.2.16). Elles forment un système couplé d'équations elliptiques non-linéaires pour $\bar{g}$ et K, respectivement une métrique riemannienne et un 2-tenseur symétrique sur une variété Σ, que l'on suppose de dimension 3 dans la suite. Quatre équations pour 12 inconnues (les 6 coefficients indépendants de $\bar{g}$ et K), il s'agit donc d'un système sous-déterminé possédant *a priori* beaucoup de solutions. Nous allons nous concentrer sur la méthode de résolution la plus utilisée : la méthode conforme. Nous renvoyons au Chapitre 7 de [CB09] pour une présentation exhaustive de la théorie des équations de contraintes et à l'article de revue [CP11].

La méthode conforme est la méthode la plus utilisée pour transformer le système (1.2.15)-(1.2.16) en un système déterminé, c'est-à-dire choisir 8 paramètres pour compenser les 8 degrés de libertés du système. L'idée principale est de fixer la classe conforme de la métrique $\bar{g}$, i.e de chercher $\bar{g}$ sous la forme

$$\bar{g} = \varphi^4 \gamma$$

où γ est une métrique riemannienne sur Σ et $\varphi > 0$ est une fonction scalaire. On peut montrer que la courbure scalaire de $\bar{g}$ est alors donnée par

$$R(\bar{g}) = \varphi^{-5} \left(8\Delta_\gamma \varphi + R(\gamma)\varphi \right).$$

où $\Delta_\gamma = \gamma^{ij} D_i D_j$ est l'opérateur de Laplace-Beltrami associé à γ. Pour définir les paramètres du tenseur K, il nous faut des objets qui se comportent bien sous une transformation conforme. Les 2-tenseurs symétriques de trace et de divergence nulle (aussi appelé TT-tenseur) en sont un exemple. Nous cherchons alors K sous la forme suivante

$$K = \varphi^{-2} \left(\sigma + L_\gamma W \right) + \frac{1}{3} \varphi^4 \tau \gamma,$$

où $\tau = \mathrm{tr}_{\bar{g}} K$ est la courbure moyenne, σ est un TT-tenseur pour γ, W est une 1-forme et L_γ est l'opérateur de Killing conforme défini par $(L_\gamma W)_{ij} = D_{(i} W_{j)} - \frac{2}{3} (\mathrm{div}_\gamma W) \gamma_{ij}$. Vérifions que ces choix de paramètres correspondent aux 8 degrés de liberté du système :

- le choix de la fonction scalaire τ compte pour un degré de liberté,
- le choix du 2-tenseur symétrique remplit *a priori* 6 degrés de liberté mais σ doit aussi vérifier $\mathrm{div}_\gamma \sigma = 0$ et $\mathrm{tr}_\gamma \sigma = 0$, c'est-à-dire 4 équations, σ correspond donc à 2 degrés de liberté,

- le choix de la métrique γ remplit *a priori* 6 degrés de liberté mais en réalité on choisit la classe conforme, c'est-à-dire γ à une fonction scalaire (strictement positive) près, on a donc 5 degrés de liberté.

Le choix des paramètres τ, σ et γ remplit donc $1+2+5=8$ degrés de liberté, et on peut alors montrer que les équations de contraintes (1.2.15)-(1.2.16) pour $(\bar{g}, K)$ se réécrivent

$$\mathrm{div}_\gamma L_\gamma W = \frac{2}{3}\varphi^6 \mathrm{d}\tau, \tag{1.2.19}$$

$$8\Delta_\gamma \varphi = R(\gamma)\varphi + \frac{2}{3}\tau^2\varphi^5 + |\sigma + L_\gamma W|^2_\gamma \varphi^{-7}, \tag{1.2.20}$$

où les inconnues sont maintenant φ et W, c'est-à-dire 4 inconnues pour les 4 équations (1.2.19)-(1.2.20). La méthode conforme permet donc de transformer les équations de contraintes en un système déterminé.

L'équation (1.2.20) est connue sous le nom d'équation de Lichnerowicz. Un moyen efficace de résoudre ce système et de choisir le paramètre τ constant sur Σ, on parle de cas CMC pour *constant mean curvature*. Ce choix découple (1.2.19) et (1.2.20) et la difficulté réside alors dans la résolution de (1.2.20). On peut considérer le cas proche de CMC, c'est-à-dire où $\frac{\mathrm{d}\tau}{\tau}$ est petit. Dans ce cas, nous renvoyons à [CBIY00] pour les résultats les plus récents et pour l'adaptation de la méthode constructive au cas des variété asymptotiquement plates.

1.2.4 Stabilité de l'espace-temps de Minkowski

1.2.4.1 Solutions explicites et stabilité en relativité générale

Il existe plusieurs solutions explicites des équations d'Einstein dans le vide (1.2.3). La plus simple est l'espace-temps de Minkowski, c'est-à-dire la variété $\mathbb{R}^{3+1}$ muni de la métrique lorentzienne

$$m = -(\mathrm{d}t)^2 + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2. \tag{1.2.21}$$

L'espace-temps de Minkowski est le cadre naturel de la relativité restreinte fondée par Poincaré et Lorentz. La métrique de Schwarzschild est une autre solution explicite. Découverte seulement quelques mois après les premières publications d'Einstein sur la relativité générale, elle s'écrit

$$g_M = -\left(1 - \frac{2M}{r}\right)(\mathrm{d}t)^2 + \left(1 - \frac{2M}{r}\right)^{-1}(\mathrm{d}r)^2 + r^2\left((\mathrm{d}\theta)^2 + \sin^2(\theta)(\mathrm{d}\phi)^2\right)$$

sur la variété $\mathbb{R}_t \times]2M, +\infty[\times \mathbb{S}^2$ et dépend d'un paramètre $M > 0$.

Une question naturelle dans le domaine des équations aux dérivées partielles est la stabilité des solutions explicites. Pour les EDPs d'évolution classiques de la physique, la formulation du problème est simple : une solution u connue est dite stable si des données initiales proches (définies sur $\{t=0\}$) donnent lieu à une solution existant pour tout temps et convergeant vers u en temps long. Pour les équations d'Einstein cette formulation ne fonctionne pas car il n'y a pas de concept de temps intrinsèque. On dit donc qu'un espace-temps donné est stable si des données initiales proches (définie sur Σ_0) donnent lieu à un développement maximal géodésiquement complet. Cette dernière notion est l'équivalent en relativité générale de "la solution existe pour tout temps".

La stabilité de l'espace-temps de Minkowski a été démontrée pour la première fois en 1993 par Christodoulou et Klainerman, voir [CK93]. Leur preuve se base sur un choix de jauge dynamique, où l'espace-temps est feuilleté par des cônes sortant et des hypersurfaces maximales. Ce résultat a été revisité par Lindblad et Rodnianski dans [LR10], où les auteurs se placent dans la jauge harmonique. Comme on l'a expliqué dans la Section 1.2.2, dans cette jauge les équations d'Einstein dans le vide se réécrivent comme un système d'équations d'ondes quasi-linéaires avec non-linéarité quadratique pour les coefficients de la métrique.

Mentionnons les travaux récents et indépendants de Klainerman et Szeftel sur la stabilité sous perturbations polarisées de la famille des métriques de Schwarzschild [KS20] et de Dafermos, Holzegel, Rodnianski et Taylor sur la stabilité en codimension finie de cette même famille [DHRT21].

1.2.4.2 Existence globale des ondes

La preuve de [LR10] se base donc sur ce qui est connu pour les équations d'ondes classiques, et en particulier sur la fameuse méthode des champs de vecteurs. Introduite par Klainerman dans [Kla85], cette méthode permet d'incorporer aux estimations d'énergie la décroissance naturelle des solutions à l'équation d'onde linéaire et homogène. Comme son nom l'indique, la méthode des champs de vecteurs se base sur l'utilisation de champs de vecteurs spéciaux, à savoir

$$\{\partial_\alpha, \quad S = t\partial_t + r\partial_r, \quad \Omega_{\alpha\beta} = x_\alpha\partial_\beta - x_\beta\partial_\alpha\}, \tag{1.2.22}$$

en lieu et place des champs de vecteurs ∂_α usuels. On note Z un élément de (1.2.22). Ces champs de vecteurs commutent avec l'opérateur $\Box$ (sauf S qui vérifie $[\Box, S] = 2\Box$) ce qui permet d'estimer $\left\|\partial Z^I u\right\|_{L^2}$ via l'inégalité d'énergie classique de l'opérateur $\Box$ si u vérifie une équation d'onde. Du point de vue de l'existence globale de solution, ces champs de vecteurs ont l'intérêt d'incorporer dans leur définition des poids, ce qui se traduit concrètement par une injection de Sobolev "dispersive", aussi appelé inégalité de Klainerman-Sobolev

$$|u| \lesssim \frac{1}{(1+s)^{\frac{n-1}{2}}\sqrt{1+|q|}} \sum_{|I|\leq \frac{n+2}{2}} \left\|Z^I u\right\|_{L^2} \tag{1.2.23}$$

où $s = t + r$ et $q = t - r$ et u est définie sur $\mathbb{R}^{n+1}$ (voir [Sog95] pour une preuve).

L'inégalité (1.2.23) remontre en particulier que la décroissance naturelle des ondes est en $t^{-\frac{n-1}{2}}$, qui est intégrable si et seulement si $n \geq 4$. Pour montrer l'existence globale de solutions à une équation d'onde non-linéaire en dimension 3 d'espace, il faut donc faire des hypothèses sur la non-linéarité. Dans [Chr86] et [Kla86], Christodoulou et Klainerman introduisent indépendamment le concept de structure nulle. Une non-linéarité quadratique $\partial u \partial u$ possède la structure nulle si elle est de la forme $\partial u \overline{\partial} u$ où $\overline{\partial} u$ est une "bonne" dérivée, c'est-à-dire tangente au cône de lumière dans l'espace-temps de Minkowski. Pour cette classe de non-linéarités, Christodoulou et Klainerman montrent qu'il y a effectivement existence globale de solutions dans $\mathbb{R}^{3+1}$ pour des données initiales petites.

Comme expliqué dans [CB00], la non-linéarité quadratique de (1.2.10) ne présente pas la structure nulle. Pour s'en sortir, Lindblad et Rodnianski utilise une version affaiblie de la structure nulle, appelée depuis leur article la structure nulle faible.

1.3 Ondes gravitationnelles

Une différence majeure entre les théories de la gravité newtonienne et einsteinienne est que dans cette dernière la gravité est dynamique, et les perturbations de la métrique peuvent se propager dans l'espace-temps. Ce sont les fameuses ondes gravitationnelles détectées en 2015. Dans cette section, nous présentons le point de vue linéaire puis non-linéaire sur les ondes gravitationnelles. L'auteur conseille la lecture de l'article de revue [FH05].

1.3.1 Linéarisation des équations d'Einstein

Les équations d'Einstein dans le vide (1.2.3) sont non-linéaires et admettent comme solution triviale la métrique de Minkowski (1.2.21). D'un point de vue physique comme mathématique, il est donc naturel de linéariser (1.2.3) autour de (1.2.21). Dans cette section, nous présentons cette procédure en s'inspirant des Chapitres 18 et 35 de [MTW73] et de la Section 4.4 de [Wal84].

Pour linéariser (1.2.3) autour de (1.2.21), on considère une métrique de la forme $g = m+h$ où m est donné par (1.2.21) et h est une petite perturbation. On calcule le tenseur de Ricci de g et on ne garde que les termes linéaires en h et ses dérivées. Dans ce régime, les symboles de Christoffel (cf (1.1.4)) sont donnés par

$$\begin{aligned}\Gamma^{\mu}_{\alpha\beta} &= \frac{1}{2}m^{\mu\nu}\left(\partial_\alpha h_{\beta\nu} + \partial_\beta h_{\alpha\nu} - \partial_\nu h_{\alpha\beta}\right)\\ &= \frac{1}{2}\left(\partial_\alpha h^{\mu}_{\beta} + \partial_\beta h^{\mu}_{\alpha} - \partial^\mu h_{\alpha\beta}\right)\end{aligned}$$

où on a adopté la convention de monter et descendre les indices par rapport à m (et non par rapport à g). Le tenseur de Ricci (cf (1.1.8)) est alors donné par

$$\begin{aligned}R_{\mu\nu} &= \partial_\alpha \Gamma^{\alpha}_{\mu\nu} - \partial_\mu \Gamma^{\alpha}_{\nu\alpha}\\ &= \frac{1}{2}\left(\partial_{(\mu}\partial_\alpha h^{\alpha}_{\nu)} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu \mathrm{tr}h\right)\end{aligned}$$

où $\mathrm{tr}h = h^{\alpha}_{\alpha}$ et $\Box = m^{\alpha\beta}\partial_\alpha\partial_\beta$ est l'opérateur d'onde usuel. On introduit les notations usuelles $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}m_{\mu\nu}\mathrm{tr}h$ et

$$f_\nu = \partial_\alpha h^{\alpha}_{\nu} - \frac{1}{2}\partial_\nu \mathrm{tr}h = \partial^\alpha \bar{h}_{\alpha\nu}.$$

Les équations d'Einstein dans le vide linéarisées prennent alors la forme

$$-\Box\bar{h}_{\mu\nu} + \partial_{(\mu}f_{\nu)} - m_{\mu\nu}\partial^\alpha f_\alpha = 0. \tag{1.3.1}$$

Pour simplifier (1.3.1), nous allons effectuer un choix de jauge. Un difféomorphisme est infinitésimalement engendré par un champs de vecteur ξ_μ et c'est la dérivée de Lie de la métrique *background* m qui représente la perturbation au niveau linéaire. On peut donc remplacer $\bar{h}_{\mu\nu}$ par $\bar{h}_{\mu\nu} + \partial_{(\mu}\xi_{\nu)} - m_{\mu\nu}\partial^\alpha\xi_\alpha$ dans (1.3.1) pour obtenir la même équation avec f_ν remplacé par $f_\nu + \Box\xi_\nu$. Quitte à résoudre une équation d'onde sur ξ_ν, on peut donc supposer que $f_\nu = 0$ (jauge de Lorentz, par analogie avec l'électromagnétisme) et les équations d'Einstein dans le vide linéarisées deviennent

$$\Box\bar{h}_{\mu\nu} = 0, \tag{1.3.2}$$

$$\partial^\alpha\bar{h}_{\alpha\nu} = 0, \tag{1.3.3}$$

où (1.3.3) est la condition de jauge de Lorentz. Cette dernière ne détermine pas totalement le système de coordonnées, car elle est invariante si on remplace $\bar{h}_{\mu\nu}$ par $\bar{h}_{\mu\nu} + \partial_{(\mu}\xi_{\nu)} - m_{\mu\nu}\partial^\alpha\xi_\alpha$ avec $\Box\xi_\nu = 0$. Remarquons que dans [AJOR21] les auteurs obtiennent les équations (1.3.2) et (1.3.3) en linéarisant l'action d'Einstein-Hilbert (défini en (1.2.6)) plutôt que le tenseur d'Einstein.

1.3.1.1 Ondes planes

Toute solution de (1.3.2) peut se décomposer sur la base des ondes planes monochromatiques, i.e les solutions de (1.3.2)-(1.3.3) de la forme

$$\bar{h}_{\mu\nu} = A_{\mu\nu}\cos(k_\alpha x^\alpha) \tag{1.3.4}$$

où A et k sont constants et vérifient $k_\alpha k^\alpha = 0$ et $A_{\mu\nu}k^\nu = 0$ (conditions équivalentes à (1.3.2) et (1.3.3) respectivement). La perturbation (1.3.4) représente une onde gravitationnelle de fréquence

$$k^t = \left(k_x^2 + k_y^2 + k_z^2\right)^{\frac{1}{2}}$$

se déplaçant à la vitesse de la lumière dans la direction (k_x, k_y, k_z). Pour une onde de la forme (1.3.4) la liberté résiduelle de la jauge de Lorentz se traduit par le choix de

$$\xi_\nu = C_\nu \sin(k_\alpha x^\alpha)$$

avec C constant qui vérifie bien $\Box\xi_\nu = 0$ car $k_\alpha k^\alpha = 0$. Si on revient à $\bar{h}_{\mu\nu}$, cela revient à pouvoir remplacer $A_{\mu\nu}$ par $A_{\mu\nu} + C_{(\mu}k_{\nu)} - m_{\mu\nu}k^\alpha C_\alpha$. On se débarasse de cette liberté résiduelle dans la section suivante où l'on définit la jauge TT.

1.3.1.2 La jauge TT

Nous allons montrer qu'on peut toujours choisir C_α tels que $\bar{h}_{t\mu} = 0$ et $\mathrm{tr}\bar{h} = 0$. On parle alors de jauge TT pour les mots anglais *transverse* et *traceless*. Si $\bar{h}$ est l'onde plane donnée par (1.3.4) alors pour A fixé on cherche C_α tel que $A_{\mu\nu} + C_{(\mu}k_{\nu)} - m_{\mu\nu}k^\alpha C_\alpha$ n'ait aucun coefficient temporel et soit de trace nulle. C'est équivalent à choisir

$$A_{t\nu} + C_t k_\nu + C_\nu k_t - m_{t\nu}k^\alpha C_\alpha = 0, \tag{1.3.5}$$
$$\mathrm{tr}A - 2k^\alpha C_\alpha = 0. \tag{1.3.6}$$

On injecte (1.3.6) dans (1.3.5) puis on prends $\nu = i$ et $\nu = t$ dans (1.3.5). On obtient

$$C_t = -\frac{1}{2k_t}\left(\frac{1}{2}\mathrm{tr}A + A_{tt}\right) \quad \text{et} \quad C_i = -\frac{1}{k_t}\left(C_t k_i + A_{ti}\right). \tag{1.3.7}$$

On peut donc supposer que le tenseur amplitude dans (1.3.4) est sous forme TT, c'est-à-dire purement spatial et de trace nulle. Montrons qu'on ne peut pas ajouter de nouvelles contraintes sur $\bar{h}_{ij}$ par un nouveau choix de coordonnées (autrement dit, nous avons épuisé nos différents choix de jauge possibles). En effet, un calcul direct montre que

$$R_{itjt} = -\frac{1}{2}\partial_t^2\bar{h}_{ij}. \tag{1.3.8}$$

Comme le tenseur de Riemann de la gravité linéarisé est un invariant de jauge, (1.3.8) montre que les composantes $\bar{h}_{ij}$ le sont aussi.

Combien de degrés de liberté le tenseur $\bar{h}$ défini par (1.3.4) possède-t-il encore ? Il possède *a priori* 10 composantes indépendantes, et vérifie $\bar{h}_{\mu\nu}k^\mu = 0$, $\bar{h}_{t\mu} = 0$ et $\mathrm{tr}\bar{h} = 0$, c'est-à-dire 8 relations. Une onde gravitationnelle possède donc 2 degrés de libertés, qui s'interprètent comme deux modes de polarisation. Pour simplifier, imaginons une onde se déplaçant dans la direction z, c'est-à-dire $k_x = k_y = 0$ et $k_z \neq 0$. Si la jauge TT est vérifiée, les seules composantes non-nulles de A sont alors $A_{xx} = -A_{yy}$ et $A_{xy} = A_{yx}$. L'onde est donc combinaison linéaires des deux tenseurs de polarisation e_+ et $e_\times$ défini par

$$e_+ = \mathrm{d}x \otimes \mathrm{d}x - \mathrm{d}y \otimes \mathrm{d}y \quad \text{et} \quad e_\times = \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x.$$

Les symboles + et × font référence à la façon dont une onde proportionnelle à e_+ ou $e_\times$ déforme les géodésiques de l'espace-temps, et donc un potentiel détecteur (voir la Section 35.6 de [MTW73]).

1.3.2 Au-delà de l'approximation linéaire

Linéariser les équations d'Einstein est une étape nécessaire pour en saisir certaines des propriétés les plus fondamentales, d'un point de vue physique et mathématique. L'existence d'ondes gravitationnelles obtenues par ce processus est un des résultats les plus spectaculaires des travaux d'Einstein. Cependant, cette description linéaire n'est pas totalement satisfaisante, à la fois d'un point de vue mathématique et physique.

1.3.2.1 Les limites du linéaire

D'un point de vue physique, l'approximation linéaire n'est valide que si les effets de la gravité sont faibles (ce qui se traduit par le fait que $\left|\partial^k h\right| \ll 1$). Or, ces effets ne sont pas toujours faibles et il existe des phénomènes où l'approximation linéaire n'est absolument pas valide. On peut penser par exemple à la fusion de deux trous noirs, événement qui est d'ailleurs une des sources principales d'ondes gravitationnelles. Dans ce cas, l'approximation linéaire peut donc modéliser la propagation de l'onde loin des deux trous noirs mais elle ne peut décrire sa création.

Par ailleurs, l'approximation linéaire ne peut décrire l'énergie de l'onde gravitationnelle. Le concept d'énergie en relativité générale est compliqué à définir, car la métrique de l'espace-temps joue à la fois le rôle d'un arrière-plan ou *background* et d'un objet dynamique. Or on voudrait associer une énergie uniquement à l'aspect dynamique, évoluant par rapport à un *background* fixe. Dans le cas de l'approximation linéaire, h joue le rôle de la perturbation dynamique mais on s'attend à ce que l'énergie soit quadratique en h, et doit donc être négligée.

Par ailleurs, quels seraient les effets de cette énergie sur le *background* ? Dans le cas de l'approximation linéaire de la section précédente, ces effets sont nuls car le *background* minkowskien ne peut pas voir l'onde.

D'un point de vue mathématique, il est évidemment important de considérer les équations (1.2.3) complètes. Les non-linéarités ne sont pas uniquement des perturbations négligeables, comme l'illustre la question de la stabilité de l'espace-temps de Minkowski décrite à la Section 1.2.4.1. Dans le cas linéaire, il s'agit de montrer la globalité des solutions de (1.3.2), ce qui nécessite quelques lignes, alors que la preuve de Christodoulou et Klainerman de la stabilité non-linéaire est un livre de 500 pages.

Pour toutes ces raisons, il nous faut un formalisme capable de décrire des phénomènes gravitationnels forts où l'énergie est clairement définie et impacte significativement le *background* selon un schéma

$$G_{\mu\nu}\,(\mathrm{BG}+\mathrm{onde})=0 \iff G_{\mu\nu}\,(\mathrm{BG})=\text{tenseur énergie impulsion associé à l'onde.}$$

De plus, ce formalisme doit pouvoir s'implémenter pour les équations non-linéaires (1.2.3).

1.3.2.2 Construction de solutions radiatives approchées à la Choquet-Bruhat

On peut retracer la recherche d'un tel formalisme à [BH64], où les auteurs considèrent un processus de moyenne sur plusieurs longueurs d'ondes. L'idée centrale est de considérer que la perturbation $h_{\mu\nu}$ est petite mais que ses dérivées premières ne le sont pas. Voir aussi les articles [Isa68a, Isa68b]. Une version plus rigoureuse mathématiquement et qui ne fait appel à aucun procédé de moyenne a été proposée par Choquet-Bruhat dans [CB69] (voir aussi le Chapitre 11 de [CB09]). Nous allons décrire en détail cet article, en utilisant des notations adaptées à cette thèse.

Dans [CB69], Choquet-Bruhat considère l'approximation de l'optique géométrique. Cette approche et son histoire seront détaillées dans la Section 1.5.1, mais dans le cas qui nous intéresse ici, cela veut dire qu'on considère une famille de métriques lorentziennes de la forme

$$g_\lambda(x)=g_0(x)+\lambda g^{(1)}\left(x,\frac{u_0(x)}{\lambda}\right)+\lambda^2 g^{(2)}\left(x,\frac{u_0(x)}{\lambda}\right) \tag{1.3.9}$$

où $\lambda>0$ est une petite longueur d'onde, g_0 est une métrique pour l'instant inconnue, et u_0 est une fonction scalaire jouant le rôle d'une phase. Les perturbations $g^{(1)}$ et $g^{(2)}$ sont fonctions de la variable d'espace-temps x et de la phase $\frac{u_0(x)}{\lambda}$. Dans la suite et pour alléger les notations on écrira plutôt $g^{(i)}\left(\frac{u_0}{\lambda}\right)$. Dans la limite haute-fréquence où λ tend vers 0, g_λ est proche de g_0 mais ce n'est pas le cas pour ses dérivées. En effet on a

$$\partial g_\lambda=\partial g_0+\partial u_0\partial_\theta g^{(1)}+O\left(\lambda\right) \tag{1.3.10}$$

où ∂_θ signifie la dérivée par rapport à la phase $\frac{u_0}{\lambda}$ (convention que l'on utilise dans toute cette thèse). Pour assurer une limite faible quand λ tend vers 0 à (1.3.10) on suppose de plus que $g^{(1)}$ est une fonction périodique de $\frac{u_0}{\lambda}$, le lemme de Riemann-Lebesgue nous assurant alors que ∂g_λ converge faiblement vers ∂g_0. Cette dernière propriété est l'analogue rigoureux des procédés de moyenne envisagés dans [BH64, Isa68a, Isa68b].

L'objectif de Choquet-Bruhat est de trouver des conditions sur les différents termes de (1.3.9) impliquant

$$R_{\mu\nu}(g_\lambda)=O\left(\lambda\right). \tag{1.3.11}$$

Le tenseur de Ricci contenant au plus des dérivées secondes de la métrique, le tenseur $R_{\mu\nu}(g_\lambda)$ contient *a priori* des termes d'ordre λ^{-1} et λ^0 qui doivent être annulés pour obtenir (1.3.11). Le résultat de Choquet-Bruhat peut alors s'énoncer sous la forme suivante.

- **Pour annuler les termes d'ordre $\frac{1}{\lambda}$ de $R_{\mu\nu}(g_\lambda)$.**
  - La phase u_0 doit satisfaire l'équation eikonale pour la métrique g_0, i.e

$$g_0^{-1}(\mathrm{d}u_0,\mathrm{d}u_0)=0.$$

  - La perturbation $g^{(1)}$ doit satisfaire $\mathrm{Pol}\left(g^{(1)}\right) = 0$ où Pol est le tenseur de polarisation défini par

$$\mathrm{Pol}_\alpha(T) = g_0^{\mu\nu}\left(\partial_\mu u_0 T_{\nu\alpha} - \frac{1}{2}\partial_\alpha u_0 T_{\mu\nu}\right). \tag{1.3.12}$$

- **Pour annuler les termes d'ordre λ^0 de $R_{\mu\nu}(g_\lambda)$.**
  - Le tenseur $g^{(1)}$ doit satisfaire une équation de transport le long des rayons de la phase u_0

$$2\partial^\alpha u_0 \mathbf{D}_\alpha g^{(1)}_{\mu\nu} + (\mathbf{D}^\alpha \mathbf{D}_\alpha u_0) g^{(1)}_{\mu\nu} = 0 \tag{1.3.13}$$

    où les indices sont montés avec la métrique g_0 et où $\mathbf{D}$ est sa dérivée covariante.
  - La perturbation $g^{(2)}$ doit vérifier des conditions de polarisation de la forme

$$\mathrm{Pol}\left(g^{(2)}\right) = (g^{(1)})^2.$$

  - La métrique *background* g_0 doit vérifier

$$R_{\mu\nu}(g_0) = \tau \partial_\mu u_0 \partial_\nu u_0. \tag{1.3.14}$$

    avec $\tau > 0$ qui dépend de la perturbation $g^{(1)}$ et dont l'interprétation physique est donnée ci-dessous.

Des commentaires s'imposent sur ces conditions.

1. Les conditions de polarisation sur $g^{(1)}$ et $g^{(2)}$ traduisent la non-hyperbolicité des équations (1.2.3), qui est liée à son invariance par difféomorphisme. La condition $\mathrm{Pol}\left(g^{(1)}\right) = 0$ implique que la partie significative de l'onde, c'est-à-dire la partie tangente aux surfaces de niveaux de u_0, a deux degrés de liberté. On retrouve ainsi les caractéristiques formelles des ondes planes dans l'approximation linéaire.

2. Du point de vue de l'optique géométrique, l'équation (1.3.13) n'est pas surprenante (nous y reviendrons dans la Section 1.5.1). Ce qui l'est plus, c'est la linéarité de (1.3.13). En effet les équations d'Einstein sont fortement non-linéaires et on pourrait s'attendre à une déformation des ondes le long des géodésiques nulles. La condition de polarisation de $g^{(1)}$ et la structure particulière des non-linéarités quadratiques du tenseur de Ricci (voir Section 1.6.3.2) sont à l'origine de cette simplification. En suivant Boillat dans [Boi96], on dit que les ondes gravitationnelles de la relativité générale sont *exceptionnelles*.

3. Au vu de (1.3.14), la métrique *background* g_0 ne décrit pas un espace-temps vide mais la perte par radiation de l'énergie contenue dans l'onde, car $\tau > 0$. Le formalisme de l'optique géométrique permet donc bien de décrire l'énergie de l'onde et comment elle est perçue par le *background*.

L'article [CB69] propose donc un formalisme qui comble les lacunes de l'approximation linéaire décrites à la Section 1.3.2.1. Cependant, ce n'est qu'une construction formelle qui ne résout que partiellement les équations d'Einstein (cf (1.3.11)). En particulier, Choquet-Bruhat n'effectue aucun choix de jauge. Comme l'indique Métivier dans [Mé09], une construction de l'approximation de l'optique géométrique en relativité générale rigoureuse du point de vue de l'analyse est restée un problème ouvert.

Pour finir, mentionnons que ces techniques ont aussi été appliquées au couplage entre la relativité générale et d'autres théories physiques, par exemple au couplage Einstein-Maxwell dans [CB71] ou aux fluides relativistes dans [CBG09]. Notons aussi que l'approximation de l'optique géométrique a aussi été appliqué aux ondes gravitationnelles dans le cadre linéaire dans [AJOR21] et [HO22]. L'objectif des auteurs est de retrouver pour les ondes gravitationnelles un effet Hall géométrique similaire à celui des ondes électromagnétiques décrit dans [OJD⁺20].

1.4 La conjecture de Burnett

L'article de Choquet-Bruhat [CB69] présenté ci-dessus montre comment on peut décrire l'impact d'une onde gravitationnelle sur la métrique *background*, en allant au-delà de l'approximation linéaire. En effet, les métriques g_λ construites sont des solutions approchées des équations d'Einstein dans le vide (cf (1.3.11)) alors que la métrique *background* g_0 satisfait (1.3.14), c'est-à-dire ne décrit pas un espace-temps vide.

En quel sens la suite $(g_\lambda)_\lambda$ converge-t-elle vers g_0 quand λ tend vers 0 ? Comme $g_\lambda = g_0 + O(\lambda)$, on a convergence forte au niveau des métriques elles-mêmes, et comme on l'a expliqué ci-dessus ∂g_λ converge faiblement vers ∂g_0 (cf (1.3.10) et la discussion qui suit). La construction de Choquet-Bruhat nous indique donc qu'on ne peut pas passer à la limite dans les équations d'Einstein dans le vide pour ce type de convergence, c'est-à-dire forte pour g et faible pour ∂g. On parle de *backreaction*, et ce phénomène est rendu possible par la présence des termes quadratiques $\partial g \partial g$ dans le tenseur de Ricci (voir (1.1.8)), car limite faible et produit ne commutent pas.

Une question plus générale peut donc être posée : quels espace-temps peuvent être obtenus par ce processus ? Autrement dit, quels termes produit-on lorsqu'on passe à la limite dans le tenseur de Ricci pour ce type de convergence ? La première formulation mathématique de ces questions est due à Burnett dans un article de 1989.

1.4.1 Enoncé de la conjecture

Dans son article [Bur89], Burnett considère le cadre mathématique suivant. Il considère une suite de métrique $(g_\lambda)_\lambda$ indéxée par un petit paramètre $\lambda > 0$ telle que

1. chaque g_λ est solution des équations d'Einstein dans le vide, i.e $G_{\mu\nu}(g_\lambda) = 0$,
2. il existe une métrique g_0 telle que

$$\begin{aligned} g_\lambda &\to g_0, \quad \text{uniformément dans } L^\infty, \\ \partial g_\lambda &\rightharpoonup \partial g_0, \quad \text{faiblement dans } L^2. \end{aligned}$$

Dans cette introduction, on dira que g_λ converge vers g_0 au sens de Burnett pour signifier la convergence précédente. La question de la *backreaction* se formule alors ainsi : quelles équations d'Einstein vérifie g_0, i.e qui est $G_{\mu\nu}(g_0)$? Remarquons tout de suite que si la convergence de $(\partial g_\lambda)_\lambda$ est forte dans L^2 alors on a nécessairement $G_{\mu\nu}(g_0) = 0$. La *backreaction* n'est donc intéressante que s'il y a convergence faible sans convergence forte.

Burnett propose dans [Bur89] la réponse suivante : la métrique g_0 est solution du système Einstein-Vlasov de masse nulle (voir Section 1.2.1), i.e il existe une densité f telle que

$G_{\mu\nu}(g_0) = T_{\mu\nu}(f)$ où $T_{\mu\nu}(f)$ est défini par (1.2.4). Burnett conjecture aussi une sorte de réciproque : toute solution du système Einstein-Vlasov de masse nulle est limite au sens de Burnett d'une suite de solutions à (1.2.3). Cette double conjecture décrit complètement l'adhérence pour la topologie de Burnett de l'ensemble des solutions à (1.2.3).

La première partie de la conjecture est appelée la conjecture *directe*, tandis que la deuxième partie est appelée la conjecture *indirecte*. Elles sont de nature très différente. Démontrer la conjecture directe nécessite de considérer une suite quelconque vérifiant les hypothèses ci-dessus et de démontrer les propriétés formelles de la limite. Pour démontrer la conjecture indirecte, on se fixe une solution g_0 du système Einstein-Vlasov de masse nulle et on construit une suite de solutions à (1.2.3) convergeant au sens de Burnett vers g_0.

Bien qu'elles constituent deux problèmes mathématiques *a priori* distincts, les conjectures de Burnett directe et indirecte sont liées. En effet, pour démontrer un théorème sur la conjecture directe, on suppose donnée la suite $(g_\lambda)_\lambda$ convergeant au sens de Burnett vers g_0. Or, il n'est pas clair que des solutions aussi pathologiques aux équations d'Einstein existent, et démontrer la conjecture indirecte donne des exemples explicites. Dans la section suivante, nous présentons la littérature sur les conjectures de Burnett, en mettant l'accent sur la conjecture indirecte, car cette thèse traite de la construction explicite de suites $(g_\lambda)_\lambda$.

1.4.2 Travaux sur la conjecture

La première avancée sur la conjecture de Burnett directe a été effectuée par Green et Wald dans [GW11], où ils démontrent que le tenseur d'Einstein de la limite g_0 est de trace nulle et vérifie la condition d'énergie faible, c'est-à-dire $G_{\mu\nu}(g_0)X^\mu X^\nu \geq 0$ pour tout X champs de vecteurs de type temps. Leur motivations viennent de la cosmologie car ils s'intéressent aux effets des inhomogénéités locales de la structure de l'espace-temps sur son évolution globale (voir aussi [BCE$^+$15] et [GW15] pour une discussion du rôle de la *backreaction* en cosmologie).

Bien qu'en accord avec la conjecture de Burnett, car le tenseur $T_{\mu\nu}(f)$ décrit plus haut est effectivement sans trace (les particules décrites étant de masses nulles) et vérifie la condition d'énergie faible, les résultats de Green et Wald sont loin de la conjecture de Burnett complète. Dans la suite de cette section, nous présentons les résultats mathématiques sur cette dernière. Comme nous allons le voir, l'enjeu principal de ces travaux est de trouver un cadre mathématique où une bonne théorie de Cauchy peut être construite pour les équations d'Einstein dans le vide. En effet, les suite $(g_\lambda)_\lambda$ sont nécessairement singulières en un certain sens, ce qui permet le défaut de convergence forte de $(\partial g_\lambda)_\lambda$. Le cadre fonctionnel et géométrique choisi doit donc autoriser l'existence de telles solutions à (1.2.3).

1.4.2.1 La symétrie $\mathbb{U}(1)$

Les premiers travaux mathématiques sur la conjecture de Burnett sont obtenus par Huneau et Luk dans une série d'articles que nous présentons ici. Ils y démontrent la conjecture de Burnett pour des espaces-temps satisfaisant la symmétrie $\mathbb{U}(1)$, c'est-à-dire si l'espace-temps admet un champs de Killing de type espace (on dit alors que l'espace-temps est invariant par translation dans une direction spatiale). Cette symétrie a été introduite par Moncrief dans [Mon86]. On peut montrer (voir Appendice 7 de [CB09]) qu'une métrique ${}^{(4)}g$ en symmétrie $\mathbb{U}(1)$ est de la forme

$${}^{(4)}g = e^{-2\varphi}g + e^{2\varphi}(\mathrm{d}x^3 + A_\alpha \mathrm{d}x^\alpha)^2,$$

où φ est une fonction scalaire, g est une métrique en $2+1$ et A_α est une 1-forme. Les équations d'Einstein dans le vide pour ${}^{(4)}g$ se réécrivent alors

$$\begin{cases} \Box_g \varphi = -\frac{1}{2} e^{-4\varphi} \partial^\rho \omega \partial_\rho \omega \\ \Box_g \omega = 4 \partial^\rho \omega \partial_\rho \varphi \\ R_{\mu\nu}(g) = 2\partial_\mu \varphi \partial_\nu \varphi + \frac{1}{2} e^{-4\varphi} \partial_\mu \omega \partial_\nu \omega \end{cases} \tag{1.4.1}$$

où ω est le potentiel twist associé à $\partial_\mu A_\nu - \partial_\nu A_\mu$. Dans leurs travaux sur la conjecture de Burnett en symétrie $\mathbb{U}(1)$, Huneau et Luk se placent de plus dans une jauge elliptique pour la métrique g. Cette jauge n'existe qu'en dimension $2+1$ et permet de réécrire la troisième équation du système précédent comme un système elliptique pour les coefficients de g, i.e de la forme

$$\Delta g = (\partial g)^2 + (\partial \varphi)^2 + (\partial \omega)^2 \tag{1.4.2}$$

où Δ est la laplacien usuel sur $\mathbb{R}^2$.

Dans [HL19], Huneau et Luk démontrent la conjecture directe en symétrie $\mathbb{U}(1)$. Leur preuve s'appuie sur les mesures de défaut micro-locales introduites dans [Gér91] et sur un mécanisme de compacité par compensation trilinéaire. La métrique limite g_0 et la densité associée sont des solutions au sens des mesures du système Einstein-Vlasov de masses nulles. Ce résultat a récemment été amélioré dans [GdC21] où les auteurs utilisent moins de dérivées de la suite $(g_\lambda)_\lambda$.

Dans [HL18b] et [HL], Huneau et Luk démontrent la conjecture indirecte en symétrie $\mathbb{U}(1)$. Dans l'article [HL18b], ils démontrent que toute solution du système Einstein-poussières nulles est limite au sens de Burnett de solutions de (1.2.3). Le système Einstein-poussières nulles est un cas particulier du système Einstein-Vlasov de masse nulle, où la densité f est une somme finie de mesures de Dirac supportées sur des hypersurfaces nulles, ces dernières étant les surfaces de niveaux de phases $u_{\mathbf{A}}$ pour $\mathbf{A}$ indice dans un ensemble fini $\mathcal{A}$. Plus précisément, la métrique en $2+1$ limite vérifie (comparé à (1.4.1) le champs ω a disparu car c'est le cas considéré dans [HL18b]) :

$$\begin{cases} \Box_{g_0} \varphi_0 = 0 \\ R_{\mu\nu}(g_0) = 2\partial_\mu \varphi_0 \partial_\nu \varphi_0 + \sum_{\mathbf{A}\in\mathcal{A}} F_{\mathbf{A}}^2 \partial_\mu u_{\mathbf{A}} \partial_\nu u_{\mathbf{A}} \\ g_0^{-1}(\mathrm{d}u_{\mathbf{A}}, \mathrm{d}u_{\mathbf{A}}) = 0 \\ 2 g_0^{\alpha\beta} \partial_\alpha u_{\mathbf{A}} \partial_\beta F_{\mathbf{A}} + (\Box_{g_0} u_{\mathbf{A}}) F_{\mathbf{A}} = 0 \end{cases}. \tag{1.4.3}$$

La suite $(\varphi_\lambda, g_\lambda)_\lambda$ est définie par des ansatz haute-fréquences multiphasiques similaires à celui envisagé par Choquet-Bruhat dans [CB69], par exemple pour φ_λ on a

$$\varphi_\lambda = \varphi_0 + \sum_{\mathbf{A}\in\mathcal{A}} \lambda F_{\mathbf{A}} \cos\left(\frac{u_{\mathbf{A}}}{\lambda}\right) + \sum_{\mathbf{A}\in\mathcal{A}} \lambda^2 \varphi_{\mathbf{A}} \left(\frac{u_{\mathbf{A}}}{\lambda}\right) + \mathcal{E}_\lambda$$

où $\mathcal{E}_\lambda$ est un reste borné par λ^2 dans H^1 et où $\varphi_{\mathbf{A}}$ est une somme finie d'harmoniques qu'on ne détaille pas ici. Des ansatz similaires sont définis pour chaque composante de la métrique. En particulier on obtient $\left\|\partial^2 g_\lambda\right\|_{L^2} \sim \frac{1}{\lambda}$. Les normes de forte régularité de $(\varphi_\lambda, g_\lambda)$ explosent donc dans la limite haute-fréquence $\lambda \to 0$. Pour gérer cette explosion, les auteurs s'appuient sur un résultat d'existence locale en faible régularité pour les équations d'Einstein en symétrie $\mathbb{U}(1)$ et en jauge elliptique obtenue dans [HL18a]. Le régime de régularité supposé dans [HL18a] est cohérent avec le caractère haute-fréquence car la petitesse est uniquement requise dans les normes $W^{1,\infty}$. La jauge elliptique joue ici un rôle crucial car le caractère elliptique des équations vérifiées par la métrique (cf (1.4.2)) permet de gagner en régularité. Elle permet donc de travailler sous le régime de régularité où les équations (1.2.3) sont bien posées, c'est-à-dire où les normes H^2 des coefficients de la métrique sont contrôlées, résultat célèbre de Klainerman, Rodnianski et Szeftel (voir [KRS15]).

Pour atteindre la conjecture de Burnett indirecte complète, Huneau et Luk approchent dans l'article à venir [HL] toute solution de Einstein-Vlasov de masse nulle par des poussières nulles, qui elles-mêmes sont approchées par des solutions de (1.2.3) dans le même esprit que dans [HL18b]. Pour pouvoir faire tendre le nombre de directions d'oscillations vers l'infini, c'est-à-dire le cardinal de $\mathcal{A}$, un résultat d'existence pour (1.2.3) en symétrie $\mathbb{U}(1)$ et jauge elliptique avec petitesse dans $W^{1,4}$ de l'auteur de cette thèse est utilisé (voir [Tou21]).

1.4.2.2 Le double feuilletage nul

Dans [LR20], Luk and Rodnianski s'intéressent à la conjecture de Burnett du point de vue du problème de Cauchy caractéristique. Dans ce formalisme (voir [Luk12] pour la construction de solutions locales), on résout les équations d'Einstein en dimension $3+1$ en prescrivant des données initiales sur deux cônes nuls H_0 et $\underline{H}_0$, respectivement sortant et rentrant, et s'intersectant en une 2-sphère $S_{0,0}$ de type espace (voir la Figure 1.1). Un des intérêts de cette formulation est que les équations de contraintes y sont de type transport et donc sont plus simples à résoudre. Cela permet de construire des données initiales au comportement dynamique intéressant, par exemple formant des surfaces piégées (voir le monumental [Chr09]).

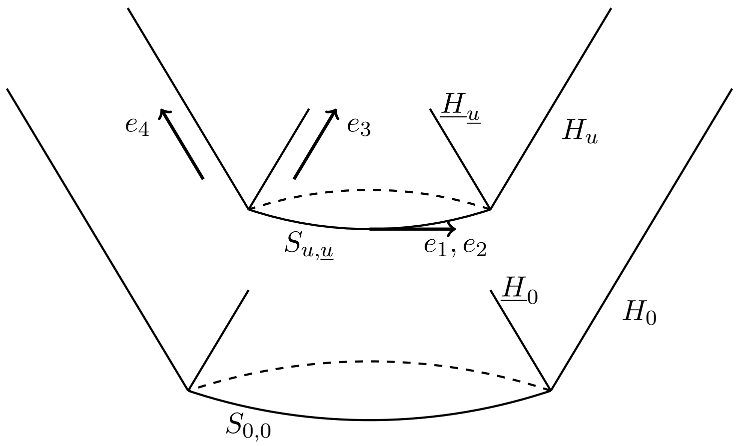

Figure 1.1: Le double feuilletage nul

Dans cette formulation, on feuillette l'espace-temps par une famille de cônes nuls émanant de H_0 et $\underline{H}_0$ et notés H_u et $\underline{H}_{\underline{u}}$ (qui sont les surfaces de niveaux de deux solutions de l'équation eikonale u et $\underline{u}$) et s'intersectant en des 2-sphères $S_{u,\underline{u}}$. On définit alors un repère nul (e_1, e_2, e_3, e_4) tel que e_3 (resp. e_4) est tangent à $\underline{H}_{\underline{u}}$ (resp H_u) et (e_1, e_2) est tangent à $S_{u,\underline{u}}$. On parle de *double feuilletage nul*, voir la Figure 1.1. Comme on peut le lire dans le Chapitre 3 de [KN03], les équations (1.2.3) se réécrivent alors comme un ensemble d'équations de transport le long des cônes pour les coefficients de Ricci $\Gamma = g(D_{e_\mu} e_\nu, e_\sigma)$ et les coefficients de courbure nuls $\Psi = R(e_\mu, e_\nu, e_\alpha, e_\beta)$. Cet ensemble d'équations compliquées est formé des équations de structure nulles pour Γ et des équations de Bianchi pour Ψ qui ont la forme suivante :

$$\begin{aligned} \not\nabla_{3,4}\Gamma &= \Gamma(1+\Gamma) + \Psi, \\ \not\nabla_{3,4}\Psi &= \not\nabla_{1,2}\Psi + \Gamma(1+\Psi). \end{aligned}$$

Dans [LR17], Luk et Rodnianski construisent des solutions locales au problème de Cauchy caractéristique telles que les dérivées de la métrique le long des directions nulles sont uniquement dans L^2, ce qui est compensé par une forte régularité des dérivées de la métrique de long des 2-sphères (on parle de régularité angulaire). Ce résultat d'existence locale est utilisé dans [LR20] pour attaquer la conjecture de Burnett en $3+1$ sans hypothèses de symétrie mais avec une forte régularité angulaire. Dans cet article, les auteurs démontrent la conjecture de Burnett directe et l'usage du résultat de [LR17] a plusieurs conséquences :

- Leur construction autorise un défaut de convergence forte par *concentration*, car la théorie locale de [LR17] requiert uniquement que la suite $(\partial g_\lambda)_\lambda$ soit bornée. C'est à comparer au résultats de [HL19, GdC21] qui requièrent tous deux un comportement de type *oscillation*, c'est-à-dire $\left|\partial^k g_\lambda\right| \lesssim \lambda^{1-k}$ pour $k = 0, 1, 2$. L'article [LR20] est ainsi le premier à produire de la *backreaction* par oscillation et concentration.

- La théorie locale de [LR17] n'autorise l'irrégularité de la métrique que dans les deux directions nulles. Par conséquent, le tenseur énergie-impulsion de la métrique limite décrit uniquement une famille de deux poussières nulles, et non une solution générale du système Einstein-Vlasov de masse nulle.

L'article [LR20] prouve aussi la conjecture de Burnett indirecte pour des poussières nulles potentiellement solutions au sens des mesures, ce qui diffère largement de l'usage des ansatz haute-fréquences dans [HL18b] ou dans cette thèse qui exige une très forte régularité de la solution *background*.

1.5 Les approximations haute-fréquences en analyse

Dans les sections précédentes, nous avons vu comment les solutions haute-fréquences sont utilisées en relativité générale pour modéliser la propagation non-linéaire des ondes gravitationnelles (Section 1.3.2) ou pour attaquer la conjecture de Burnett (Section 1.4). Dans cette section, nous évoquons deux domaines de l'analyse où les approximations ou solutions haute-fréquences sont utiles.

1.5.1 Optique géométrique

Nous commençons par présenter l'optique géométrique. Ce domaine est en réalité *défini* par le fait d'approcher des solutions par des approximations haute-fréquences. En ce sens, on peut interpréter cette thèse comme l'application de l'optique géométrique au problème de Cauchy en relativité générale, dans la continuité de [CB69]. Nous suivons la présentation faite dans [Mé09], voir aussi le livre [Rau12] (en particulier son Chapitre 7).

L'optique géométrique est le domaine des mathématiques et de la physique qui étudie la propagation d'ondes dans des systèmes hyperboliques linéaires ou non-linéaires provenant historiquement de l'optique, où le modèle le plus basique est constitué des équations de Maxwell que l'on peut coupler à un milieu. De manière générale, on considère des systèmes quasi-linéaires de la forme

$$A_0(a,u)\partial_t u + \sum_{j=1}^{d} A_j(a,u)\partial_j u = F(a,u) \tag{1.5.1}$$

où a décrit un ensemble de paramètres, $u : \mathbb{R}_t \times \mathbb{R}^d \longmapsto \mathbb{R}^N$ est une fonction vectorielle et où les A_μ sont des matrices $N \times N$ telles qu'il existe une matrice $S(a,u)$ lisse par rapport à ses arguments et telles que $S(a,u)A_0(a,u)$ est définie positive et $S(a,u)A_j(a,u)$ est auto-adjointe pour $j = 1,\dots,d$. On dit alors que (1.5.1) est symétrisable et on peut résoudre le problème de Cauchy local dans H^s si $s > \frac{d}{2}+1$, l'outil principal étant une estimation d'énergie *a priori*. Pour des systèmes tels que (1.5.1), l'optique géométrique cherche des solutions de la forme

$$u(t,x) = \varepsilon^p U\left(t, x, \frac{\varphi(t,x)}{\varepsilon}\right) \tag{1.5.2}$$

où $\varepsilon > 0$ est petit, la phase φ vérifie l'équation eikonale associée à (1.5.1) et l'amplitude U est transportée le long des rayons, c'est-à-dire le long des hypersurfaces de niveaux de φ. L'idée derrière (1.5.2) est l'interaction de deux échelles de variations différentes.

1.5.1.1 Le cas linéaire et la méthode WKB

Le cas linéaire correspond à la situation où A_0 et A_J dans (1.5.1) ne dépendent pas de u et où F est une fonction linéaire de u. Il a été étudié par Lax dans [Lax57] grâce à la méthode WKB qui, contrairement à la transformée de Fourier, peut s'étendre aux problèmes non-linéaires.

L'idée centrale de la méthode WKB (qui tire son nom des physiciens Wentzel, Kramers et Brillouin) est de développer formellement l'amplitude U de (1.5.2) et de considérer des solutions de la forme

$$u_\varepsilon(t,x) = e^{i\frac{\varphi(t,x)}{\varepsilon}}\sigma_\varepsilon(t,x) \quad \text{où} \quad \sigma_\varepsilon(t,x) \sim \sum_{n\geq 0} \varepsilon^n \sigma_n(t,x). \tag{1.5.3}$$

On considère le cas idéalisé où les matrices A_μ ne dépendent d'aucun paramètres et où $F = 0$. On insère alors l'ansatz (1.5.3) dans l'équation et on identifie les puissances de ε. On obtient alors la hiérarchie d'équations suivante

$$\mathcal{L}(\mathrm{d}\varphi)\sigma_0 = 0, \tag{1.5.4}$$

$$i\mathcal{L}(\mathrm{d}\varphi)\sigma_{n+1} + L(\partial)\sigma_n = 0, \tag{1.5.5}$$

pour $n \geq 1$ et où $\mathcal{L}(\tau,\xi) = \tau A_0 + \sum_{j=1}^d \xi_j A_j$ est le symbole de l'opérateur et $L(\partial) = A_0\partial_t + \sum_{j=1}^d A_j\partial_j$. Pour pouvoir trouver une solution à (1.5.4), la phase φ doit vérifier l'équation eikonale

$$\det(\mathcal{L}(\mathrm{d}\varphi)) = 0$$

et σ_0 doit vérifier la condition de polarisation $\sigma_0 \in \mathrm{Ker}(\mathcal{L}(\mathrm{d}\varphi))$. En résolvant la hiérarchie d'équations (1.5.4)-(1.5.5), Lax construit dans [Lax57] des solutions approchées à (1.5.1) puis des solutions exactes pour des données initiales oscillantes.

1.5.1.2 Le cas non-linéaire et la transparence

Dans le cas non-linéaire, la méthode WKB s'adapte de la manière suivante. Au lieu de supposer un comportement oscillatoire simple comme dans (1.5.3) on considère des *profils* d'oscillations, i.e on considère des solutions sous la forme

$$u_\varepsilon(t,x) = \varepsilon^p \sum_{n\geq 0} \varepsilon^n U_n\left(t, x, \frac{\varphi(t,x)}{\varepsilon}\right)$$

où les U_n sont périodiques en leur troisième variables. Ce choix est motivé par la possible interférence des ondes due aux non-linéarités qui crée des harmoniques. La première justification de l'optique géométrique dans un contexte non-linéaire remonte à [CB64].

Le choix du réel p est crucial et dépend de l'équation. Si p est trop grand, aucun effet non-linéaire ne sera perçu et les équations de transport sur les profils U_n seront linéaires, même si l'équation étudiée est non-linéaire. Abaissant progressivement la valeur de p, on augmente la force de l'onde et on atteint une valeur critique p_0 où les non-linéarités en U_0 sont de même ordre que le terme de transport associé à U_0. Cette valeur critique dépend de l'équation mais généralement on a $p_0 = \frac{1}{2}$ pour une interaction cubique (exemple : le modèle de l'oscillateur anharmonique en optique quantique) et $p_0 = 1$ pour une interaction quadratique (exemple : les équations d'Einstein étudiées dans cette thèse). Si $p = p_0$, l'équation de transport sur U_0 n'est plus linéaire, c'est le régime de *l'optique géométrique faiblement non-linéaire*. On peut alors s'attendre à des phénomènes purement non-linéaires tels que la déformation du front d'onde ou la création d'oscillations denses. Dans le cas des équations d'Euler 2D, cette situation peut résulter de l'intéraction de trois ondes comme démontré dans [JMR98].

Pour certaines équations non-linéaires, malgré le choix de $p = p_0$, l'équation de transport sur le profil principal U_0 est linéaire. C'est le phénomène de *transparence*, en général lié à des annulations dues à la structure même des non-linéarités. Il a été mis en évidence pour plusieurs systèmes d'équations venant de la physique comme par exemple le système de Maxwell-Bloch (voir [JMR00]). Dans l'article [CB69] présenté dans la Section 1.3.2.2, Choquet-Bruhat démontre de fait la transparence pour les équations d'Einstein dans le vide en construisant des solutions approchées. Ce phénomène est aussi à l'oeuvre dans [HL18b] où les auteurs construisent, en se plaçant dans une symétrie particulière (voir Section 1.4.2.1), des solutions à (1.2.3) constituées d'une superposition d'ondes se propageant linéairement. À la connaissance de l'auteur, les résultats de cette thèse sont les premiers à mettre à profit la transparence pour des solutions *exactes* de (1.2.3) *sans symétrie*.

1.5.2 Intégration convexe

L'intégration convexe est une technique d'analyse dont l'histoire remonte à Nash et son théorème de plongement isométrique C^1 (voir [Nas54]) et qui fut formalisée plus tard par Gromov dans [Gro86]. Elle a trouvé des applications spectaculaires dans la construction de solutions pathologiques aux équations de la mécanique des fluides, voir [DLS09] pour un article fondateur. Le but ici n'est pas de présenter la théorie de l'intégration convexe et des inclusions différentielles de Gromov mais plutôt de mettre en exergue les similarités entre l'usage des solutions haute-fréquences en relativité générale et en mécanique des fluides.

Rappelons les équations d'Euler incompressibles, qui décrivent l'évolution de la vitesse $v(t,x) \in \mathbb{R}^3$ d'un fluide non-visqueux subissant des forces internes de pression :

$$\partial_t v + \mathrm{div}(v \otimes v) + \nabla p = 0 \tag{1.5.6}$$

avec la contrainte d'incompressibilité $\mathrm{div}(v) = 0$ (la pression ne joue aucun rôle et sera négligée dans la suite). Pour des solutions assez régulières de (1.5.6) on a conservation de l'énergie $\int_{\mathbb{R}^3} \frac{|v(t,x)|^2}{2} \mathrm{d}x$. En revanche les techniques d'intégration convexe ont permis de construire des solutions ne conservant pas l'énergie, par exemple des solutions non-nulles à support fini en temps. C'est l'objet de la conjecture de Onsager (démontrée dans [Ise18]) : si $v \in C^\alpha_{t,x}$ avec $\alpha > \frac{1}{3}$ alors l'énergie est conservée, et pour tout $\alpha < \frac{1}{3}$ il existe des solutions faibles $v \in C^\alpha_{t,x}$ ne conservant pas l'énergie. Cette conjecture implique en particulier la non-unicité des solutions faibles de (1.5.6).

Présentons la construction de ces solutions faibles, en suivant l'introduction de [Ise17]. Elles sont obtenues par passage à la limite dans un procédé itératif où l'on construit une suite $(v_\ell)_\ell$ de sous-solutions à (1.5.6). Un des enjeux de l'intégration convexe est le choix de cet ensemble de sous-solutions. Dans le cas de (1.5.6), l'ensemble choisi est l'ensemble des solution du système Euler-Reynolds

$$\partial_t v + \mathrm{div}(v \otimes v) + \nabla p = \mathrm{div}(R), \qquad \mathrm{div}(v) = 0 \tag{1.5.7}$$

où R est un 2-tenseur symétrique et négatif. L'usage de ce système en intégration convexe date de [DLS13] et on l'obtient par exemple en régularisant une solution v de (1.5.6) (par exemple par convolution) : la vitesse approchée v_ε vérifie alors (1.5.7) avec un tenseur R mesurant la non-commutation de la régularisation et de la non-linéarité. La suite $(v_\ell)_\ell$ est alors définie ainsi : on part d'une solution (v_0, R_0) à (1.5.7) et on définit $v_1 = v_0 + V$ où V est une perturbation haute-fréquence choisi pour que v_1 résolve (1.5.7) avec $|R_1| \ll |R_0|$, et ainsi de suite pour v_2 etc. La procédure converge vers une solution (faible) de (1.5.6), et montre aussi que toute solution de (1.5.7) peut être approchée dans une topologie faible par une solution de (1.5.6).

Les similarités entre ce type de construction en mécanique des fluides et la conjecture de Burnett sont frappantes. Au-delà de l'usage commun de perturbations haute-fréquences, on peut remarquer que le système Euler-Reynolds est aux équations d'Euler ce que le système Einstein-Vlasov de masse nulle est aux équations d'Einstein dans le vide, c'est-à-dire l'adhérence faible de l'ensemble des solutions ; et l'apparition d'un membre de droite non-trivial dans les équations, $\mathrm{div}(R)$ pour la mécanique des fluides et $T_{\mu\nu}(f)$ pour la relativité générale, a dans les deux cas pour origine un défaut de commutativité entre limite faible et produit (v et ∂g jouant alors le même rôle). On note tout de même une différence de taille : le système Euler-Reynolds est sous-déterminé, alors que le système Einstein-Vlasov de masse nulle est bien posé. La question du transfert des techniques d'intégration convexe de la mécanique des fluides vers la relativité générale reste donc ouverte.

1.6 Présentation des résultats

Nous concluons ce chapitre introductif par la présentation des résultats de cette thèse. L'objectif principal de cette thèse est de construire des solutions haute-fréquences aux équations d'Einstein dans le vide en dimension $3 + 1$ sans hypothèses de symétrie et en jauge d'onde généralisée.

1.6.1 Motivations

Comme on l'a vu dans la Section 1.4.2, les constructions rigoureuses existantes de solutions haute-fréquences aux équations d'Einstein dans le vide utilisent de manière fondamentale une hypothèse de symétrie (cas des travaux de Huneau et Luk) ou le double feuilletage nul (cas des travaux de Luk et Rodnianski).

Les solutions construites dans le Chapitre 3 convergent au sens de Burnett (cf Section 1.4.1) vers une solution du système Einstein couplé à une poussière nulle. Cependant, l'intérêt principal de la jauge d'onde généralisée est qu'elle autorise *a priori* la *superposition* d'ondes, c'est-à-dire la construction de solutions oscillantes proches d'un nombre fini de poussière nulles (une solution de l'équivalent $3+1$ de (1.4.3)). Ceci n'est pas possible dans le cadre mathématique de Luk et Rodnianski dans [LR20], i.e le double feuilletage nul. En effet, la théorie de Cauchy requise par leur construction exige de compenser l'aspect singulier des solutions le long des hypersurfaces nulles par une forte régularité le long des 2-sphères. On ne peut donc imaginer ajouter une troisième direction le long de laquelle la métrique est irrégulière. La superposition d'ondes a déjà été envisagée dans [HL18b] mais avec une hypothèse de symétrie $\mathbb{U}(1)$. Le choix de la jauge d'onde généralisée permet donc *a priori* la superposition d'ondes haute-fréquences en dimension 3+1 et sans symétrie.

Le Chapitre 3, qui contient le résultat central de cette thèse, améliore donc les résultats de Huneau et Luk d'un côté en abandonnant l'hypothèse de symétrie et les travaux de Luk et Rodnianski d'un autre côté en se plaçant dans une jauge moins rigide. Il constitue aussi une version rigoureuse du point de vue du problème de Cauchy de la construction de Choquet-Bruhat dans [CB69] (cf Section 1.3.2).

Une autre motivation pour choisir la jauge d'onde généralisée est qu'elle permet d'étudier de manière naturelle la question du temps long et de la stabilité. Comme on l'a expliqué dans la Section 1.2.4, l'article [LR10] a rendu possible l'usage en relativité générale des techniques venant de l'étude en temps long des équations d'ondes non-linéaires : voir par exemple [Hun18] pour l'application à la symétrie $\mathbb{U}(1)$ ou [BFJ+21, LT20] pour l'application au système Einstein-Vlasov. En démontrant l'existence locale en temps de perturbations haute-fréquences de l'espace-temps de Minkowski en jauge d'onde généralisée, cette thèse ouvre peut-être la voie à une preuve de la stabilité non-linéaire et haute-fréquence de l'espace-temps de Minkowski. Le Chapitre 4 est aussi un premier pas dans cette direction, puisqu'on y étudie l'existence globale de solutions haute-fréquences à une équation modèle.

1.6.2 Données initiales haute-fréquences

Le Chapitre 2 est la reproduction de l'article [Tou22b]. On y construit des solutions haute-fréquences aux équations de contraintes sur la variété non-compacte $\mathbb{R}^3$. Le principal résultat de ce chapitre est donné par le théorème suivant (voir le Théorème 2.2.1 pour une version plus précise):

Théorème 1.6.1. *Soit $(\bar{g}_0, K_0, F_0, u_0)$ une solution proche de la métrique euclidienne des équations de contraintes maximale avec une poussière nulle comme source*

$$\begin{aligned} R(\bar{g}_0) - |K_0|^2_{\bar{g}_0} &= 2|\nabla u_0|^2_{\bar{g}_0} F_0^2, \\ -\mathrm{div}_{\bar{g}_0} K_0 &= |\nabla u_0|_{\bar{g}_0} F_0^2 \mathrm{d}u_0, \\ \mathrm{tr}_{\bar{g}_0} K_0 &= 0. \end{aligned}$$

Il existe une famille $(\bar{g}_\lambda, K_\lambda)$ de solutions aux équations de contraintes (1.2.15)-(1.2.16) de la forme

$$\bar{g}_\lambda = \bar{g}_0 + \lambda \bar{g}^{(1)}\left(\frac{u_0}{\lambda}\right) + \lambda^2 \bar{g}^{(2)}\left(\frac{u_0}{\lambda}\right) + \lambda^2 \bar{g}_\lambda^{(\geq 2)}, \tag{1.6.1}$$

$$K_\lambda = K^{(0)}\left(\frac{u_0}{\lambda}\right) + \lambda K^{(1)}\left(\frac{u_0}{\lambda}\right) + \lambda^2 K_\lambda^{(\geq 2)}. \tag{1.6.2}$$

La preuve du Théorème 1.6.1 est une adaptation au contexte haute-fréquence de la méthode conforme présentée à la Section 1.2.3. Les paramètres γ, τ et σ sont définis par des ansatz haute-fréquence, ainsi que les solutions φ et W des équations (1.2.19) et (1.2.20).

Comme expliqué dans la Section 1.2.2.2, les données de Cauchy pour les équations d'Einstein doivent vérifier les équations de contraintes. Le Théorème 1.6.1 est ainsi une étape nécessaire dans la construction d'espace-temps haute-fréquences solutions de (1.2.3).

La difficulté principale de ce résultat est alors formelle : les solutions construites $(\bar{g}_\lambda, K_\lambda)$ doivent être compatibles avec l'ansatz dans l'espace-temps complet. En particulier, au bas ordre en λ, la dérivée temporelle $\partial_t g$ est donné par la métrique g elle-même (car les ondes haute-fréquences vérifient des équations de transport d'ordre 1). La seconde forme fondamentale K_λ dépend donc de $\bar{g}_\lambda$ dans un certain sens et la question des données initiales n'est pas indépendante de celle de l'évolution, contrairement au cas classique présenté dans la Section 1.2.2.3 où le problème de Cauchy se résout de manière "triangulaire" (on résout d'abord les contraintes, puis le résultat est injecté dans l'évolution).

1.6.3 Existence locale en coordonnées d'ondes généralisées

Le Chapitre 3 est la reproduction de l'article [Tou22c] et il constitue le coeur de cette thèse. On y démontre le théorème suivant (voir le Théorème 3.1.2 pour une version plus précise):

Théorème 1.6.2. *Soit* (g_0, F_0, u_0) *une solution du système Einstein-poussière nulle sur* $[0,1]\times\mathbb{R}^3$ *en coordonnées d'ondes, proche de Minkowski, i.e*

$$\begin{aligned} R_{\mu\nu}(g_0) &= F_0^2\partial_\mu u_0 \partial_\nu u_0, \\ g_0^{-1}(\mathrm{d}u_0, \mathrm{d}u_0) &= 0, \\ 2g_0^{\alpha\beta}\partial_\alpha u_0 \partial_\beta F_0 + (\Box_{g_0} u_0)F_0 &= 0. \end{aligned}$$

Il existe une famille $(g_\lambda)_{\lambda\in(0,1]}$ *de métriques haute-fréquences de la forme*

$$g_\lambda = g_0 + \lambda g^{(1)}\left(\frac{u_0}{\lambda}\right) + \lambda^2 g^{(2)}\left(\frac{u_0}{\lambda}\right) + \lambda^2\mathfrak{h}_\lambda + \lambda^3 g^{(3)}\left(\frac{u_0}{\lambda}\right) \tag{1.6.3}$$

solutions des équations d'Einstein dans le vide (1.2.3) sur $[0,1]\times\mathbb{R}^3$ *en coordonnées d'ondes généralisées.*

La preuve de ce théorème suit la stratégie classique de l'optique géométrique et prolonge celle mise en place dans [CB69]. On calcule le développement en puissance de λ du tenseur de Ricci de g_λ défini par (1.6.3) pour obtenir une hiérarchie d'équations que les différents termes de (1.6.3) doivent satisfaire. Schématiquement, les ondes $g^{(1)}$ et $g^{(2)}$ satisfont des équations de transport le long des rayons de u_0 et le reste non-oscillant h_λ vérifie une équation d'onde dont l'opérateur d'onde est $\Box_{g_\lambda}$ (on ne détaille pas le rôle de $g^{(3)}$ ici). Plusieurs aspects de cette construction sont complexifiés par les particularités du tenseur de Ricci comparé à une équation d'onde classique comme (1.2.11) : la quasi-linéarité, la structure de la non-linéarité quadratique et les termes de jauges.

1.6.3.1 Les termes de jauges

Ces termes de jauges sont les H^ρ définis en (1.2.9) et sont responsable du fait que le tenseur de Ricci n'est pas un opérateur hyperbolique car les termes $g_{\rho(\mu}\partial_{\nu)}H^\rho$ contiennent aussi des dérivés secondes de la métrique (dans le Chapitre 11 de [CB09] ils sont appelés termes non-diagonaux). Si on applique ces dérivées secondes à une onde haute-fréquence on perd donc deux puissances de λ alors que l'opérateur d'onde n'en fait perdre qu'une (car le choix de la phase u_0 implique $g_\lambda^{-1}(\mathrm{d}u_0, \mathrm{d}u_0) = O(\lambda)$). Par conséquent, l'onde $g^{(i)}$ (pour $i = 1, 2$) apparaît à l'ordre λ^{i-2} sous la forme d'un terme de polarisation et à l'ordre λ^{i-1} sous la forme du terme de transport habituel. De part la structure du terme $g_{\rho(\mu}\partial_{\nu)}H^\rho$ dans (1.2.8), ces termes de polarisation sont de la forme $\partial_{(\mu}u_0\mathrm{Pol}_{\nu)}\left(g^{(i)}\right)$ (l'opérateur Pol est défini en (1.3.12)) et permettent donc d'absorber des termes orthogonaux aux surfaces de niveaux de la phase u_0.

Pour l'onde $g^{(1)}$, on retrouve le résultat de Choquet-Bruhat (cf Section 1.3.2), c'est-à-dire que $\mathrm{Pol}\left(g^{(1)}\right) = 0$. En utilisant les données initiales construites au Chapitre 2 on montre alors que l'onde $g^{(1)}$ a deux degrés de libertés, en accord avec les ondes planes de la gravité linéarisée (cf Section 1.3.1.2). Le tenseur de polarisation de l'onde $g^{(2)}$ est utilisé pour absorber la création d'harmoniques au niveau λ^0, voir la section suivante.

Les conditions de polarisation pour $g^{(1)}$ et $g^{(2)}$ sont des conditions supplémentaires que ces ondes doivent vérifier en plus des équations de transport le long des rayons. Dans [CB69], Choquet-Bruhat montre que la première condition de polarisation est propagée par l'équation de transport de $g^{(1)}$. Dans le Chapitre 3, on montre de plus que la seconde condition de polarisation est propagée par les identités de Bianchi contractées (cf Lemma 1.2.1), à la manière d'un terme de jauge d'onde. Malgré cela, les conditions de polarisation ne sont pas des conditions de jauge car ce ne sont pas des conditions sur les coordonnées mais sur les ondes $g^{(i)}$ elles-mêmes.

1.6.3.2 La non-linéarité quadratique

Comme on l'a expliqué au début de la Section 1.4, le terme $Q_{\mu\nu}(\partial g, \partial g)$ dans (1.2.8) est responsable de la *backreaction*. Il est aussi responsable de la création d'harmonique. En effet, si on se restreint à l'onde $g^{(1)}_{\alpha\beta} = \cos\left(\frac{u_0}{\lambda}\right) F^{(1)}_{\alpha\beta}$ on a

$$Q_{\mu\nu}\left(\lambda\partial g^{(1)}, \lambda\partial g^{(1)}\right) = \frac{1}{2}\left(1 + \cos\left(\frac{2u_0}{\lambda}\right)\right) Q_{\mu\nu}\left(F^{(1)}\partial u_0, F^{(1)}\partial u_0\right) + O(\lambda).$$

Le terme non-oscillant est à l'origine de la *backreaction* et le terme en $\cos\left(\frac{2u_0}{\lambda}\right)$ traduit la création d'harmonique. Le terme $Q_{\mu\nu}\left(F^{(1)}\partial u_0, F^{(1)}\partial u_0\right)$ satisfait ce que l'on appelle la *structure nulle faible polarisée*, c'est-à-dire est de la forme

$$\mathcal{E}\left(F^{(1)}, F^{(1)}\right)\partial_\mu u_0 \partial_\nu u_0$$

si $\mathrm{Pol}\left(F^{(1)}\right) = 0$ où $\mathcal{E}\left(F^{(1)}, F^{(1)}\right)$ joue le rôle d'une énergie. Cette structure est utilisée de manière cruciale dans la preuve du Théorème 1.6.2 car elle permet de contrôler la création d'harmonique par $\mathrm{Pol}\left(g^{(2)}\right)$ et de montrer ainsi que les équations de transport de $g^{(1)}$ et $g^{(2)}$ sont linéaires, malgré l'aspect fortement non-linéaire de (1.2.3). C'est le phénomène de transparence introduit dans la Section 1.5.1.2.

1.6.3.3 La quasi-linéarité

L'aspect quasi-linéaire de (1.2.3) se manifeste en coordonnées d'ondes généralisées parce que l'opérateur d'onde appliqué à g_λ dépend lui-même de g_λ. Cela a la conséquence suivante pour la hiérarchie haute-fréquence. Si on applique $\Box_{g_\lambda}$ à l'onde $\lambda g^{(1)}$, on obtient un terme proportionnel à $\frac{1}{\lambda}g_\lambda^{-1}(\mathrm{d}u_0, \mathrm{d}u_0)$. En développant ce terme on obtient, entre autres, un terme en $\lambda\mathfrak{h}_\lambda$, qui est donc absorbé par l'équation de transport de $g^{(2)}$. On obtient donc schématiquement le système couplé suivant entre $g^{(2)}$ et $\mathfrak{h}_\lambda$:

$$g_0^{\alpha\beta}\partial_\alpha u_0 \partial_\beta g^{(2)} = \mathfrak{h}_\lambda, \tag{1.6.4}$$

$$\Box_{g_\lambda}\mathfrak{h}_\lambda = \Box_{g_\lambda} g^{(2)}. \tag{1.6.5}$$

Ce système présente une perte de dérivée, et la résoudre est le principal challenge analytique du Chapitre 3. En effet des estimations d'énergie pour les équations précédentes impliquent d'un côté que $\left\|\partial g^{(2)}\right\|_{L^2} \lesssim \|\partial\mathfrak{h}_\lambda\|_{L^2}$ et de l'autre que $\|\partial\mathfrak{h}_\lambda\|_{L^2} \lesssim \left\|\partial^2 g^{(2)}\right\|_{L^2}$.

Pour regagner cette dérivée, on montre qu'on peut en réalité estimer $\Box_{g_\lambda} g^{(2)}$ par une dérivée de $g^{(2)}$. On utilise pour cela la structure du système (1.6.4)-(1.6.5) et plus particulièrement le commutateur $[L_0, \Box_{g_0}]$ que l'on estime grâce au feuilletage nul de la métrique *background* g_0. Pour contrôler la différence $\Box_{g_\lambda} - \Box_{g_0}$, on modifie le système (1.6.4)-(1.6.5) en introduisant des projecteurs spectraux $\Pi_\leq$ et $\mathrm{Id} - \Pi_\leq$, où le symbole de $\Pi_\leq$ est supporté dans $\left\{|\xi| \leq \frac{1}{\lambda}\right\}$.

1.6.4 Temps long pour une équation modèle

Le Chapitre 4 est la reproduction de l'article [Tou22a] qui a été accepté par la revue Asymptotic Analysis. Il est indépendant des deux premiers, car il ne concerne pas directement les équations de la relativité générale, mais plutôt une équation modèle, c'est-à-dire partageant certaines propriétés des équations d'Einstein tout en étant plus simple. L'objectif de cette simplification est d'investiguer le comportement en temps long des solutions haute-fréquences. L'équation modèle est la suivante:

$$\Box u = Q(\partial u, \partial u), \tag{1.6.6}$$

posée sur $\mathbb{R}_+ \times \mathbb{R}^3$ et où $\Box = -\partial_t^2 + \Delta$ est l'opérateur d'onde associé à la métrique de Minkowski et où Q est une forme quadratique vérifiant la condition nulle, c'est-à-dire s'annulant si appliqué au même vecteur nul. L'équation d'onde (1.6.6) est une simplification des équations d'Einstein écrites en coordonnées d'ondes pour deux raisons : c'est une équation semi-linéaire, car l'opérateur d'onde ne dépend pas de la solution u, et la non-linéarité vérifie la condition nulle alors que $Q_{\alpha\beta}(\partial g, \partial g)$ (cf (1.2.8)) en vérifie une version faible. Cette dernière simplification est cruciale si l'on s'intéresse à l'existence globale de solution pour (1.6.6) et a déjà été discutée à la Section 1.2.4. Le théorème principal du Chapitre 4 est le suivant (voir le Théorème 4.3.1 pour une version plus précise):

Théorème 1.6.3. *On considère des données initiales $(F_0, \varphi_0, \varphi_1)$ de taille ε. Si ε est assez petit (indépendant de λ), il existe une solution globale Φ_λ de (1.6.6) admettant la décomposition suivante*

$$\Phi_\lambda = \varphi + \lambda F \cos\left(\frac{t-r}{\lambda}\right) + \lambda^2 F_\lambda.$$

A l'image des théorèmes démontrés dans les Chapitres 2 et 3, la difficulté principale vient du caractère haute-fréquence des solutions considérées. Les données initiales étant oscillantes, on ne peut pas appliquer directement le résultat de Klainerman [Kla86] car ce dernier requiert des données initiales petites dans des espaces de Sobolev de fortes régularités. Pour contourner cette difficulté et pouvoir considérer la limite haute-fréquence, on suit la même stratégie que pour les Théorèmes 1.6.1 et 1.6.2 et on définit un ansatz haute-fréquence pour la solution Φ_λ:

$$\Phi_\lambda(t,x) = \varphi + \sum_{k=1}^{K} \lambda^k \Phi^{(k)}\left(t, x, \frac{t-r}{\lambda}\right) + h_\lambda(t,x) \tag{1.6.7}$$

où h_λ est un reste non-oscillant d'ordre λ^K. On insère l'ansatz (1.6.7) dans (1.6.6) et on dérive une série d'équations de transport pour chaque $\Phi^{(k)}$ ainsi qu'une équation d'onde pour h_λ dont la partie principale est (1.6.6). Pour démontrer le Théorème 1.6.3, il suffit donc de montrer que le système triangulaire obtenu admet une solution globale, en adaptant la méthode des champs de vecteurs de Klainerman présentée à la Section 1.2.4.

Plusieurs difficultés apparaissent lorsqu'on applique cette stratégie aux solutions haute-fréquences. Tout d'abord, les non-linéarités de (1.6.6) nécessitent de décrire dans l'ansatz la création d'harmoniques et de définir précisément comment les ondes $\Phi^{(k)}$ oscillent, c'est-à-dire dépendent de la variable $\frac{t-r}{\lambda}$. Commes ces dernières apparaissent comme source dans l'équation pour le reste h_λ, on doit aussi commuter les champs de vecteurs (1.2.22) avec l'opérateur de transport $\partial_t + \partial_r$ qui apparaît dans la hiérarchie d'équations obtenues pour les $\Phi^{(k)}$ (cf (1.6.7)). On montre que h_λ existe globalement grâce à un bootstrap où on améliore des estimations du type

$$\left\| w^{\frac{1}{2}} \partial Z^I h_\lambda \right\|_{L^2} \lesssim \varepsilon \lambda^{K-|I|}. \tag{1.6.8}$$

L'intéraction entre la dispersion naturelle des solutions et leur caractère haute-fréquence (traduit par la présence du facteur $\lambda^{K-|I|}$ dans (1.6.8)) est la principale difficulté de l'argument de bootstrap, et lors de cette étape on montre que le degré de précision de notre ansatz (1.6.7) doit vérifier $K \geq 4$.

1.6.5 Un théorème d'existence en symétrie $\mathbb{U}(1)$

Nous concluons cette introduction en présentant un résultat d'existence pour les équations d'Einstein en symétrie $\mathbb{U}(1)$, obtenu pendant le M2 de l'auteur et finalisé pendant sa thèse. Ce travail n'est pas inclus dans ce manuscrit car il n'est qu'indirectement lié aux espace-temps haute-fréquences. Il a aboutit à l'article [Tou21].

Comme on l'a expliqué dans la Section 1.4.2.1, la symétrie $\mathbb{U}(1)$ correspond à l'invariance par translation dans une direction spatiale de la variété. Les équations d'Einstein dans le vide pour ce type d'espace-temps se réécrivent comme le système Einstein couplé à une application d'onde en dimension 2+1 :

$$\begin{aligned} \Box_g \varphi &= -\frac{1}{2} e^{-4\varphi} \partial^\rho \omega \partial_\rho \omega, \\ \Box_g \omega &= 4 \partial^\rho \omega \partial_\rho \varphi, \\ R_{\mu\nu}(g) &= 2\partial_\mu \varphi \partial_\nu \varphi + \frac{1}{2} e^{-4\varphi} \partial_\mu \omega \partial_\nu \omega, \end{aligned} \tag{1.6.9}$$

où g est une métrique en $2+1$ et où φ et ω sont des fonctions scalaires. Pour résoudre ce système, on choisit de se placer dans une jauge dite elliptique, i.e de réécrire la métrique g sous la forme

$$g = -N^2 (\mathrm{d}t)^2 + e^{2\gamma} \delta_{ij} \left(\mathrm{d}x^i + \beta^i \mathrm{d}t \right) \left(\mathrm{d}x^j + \beta^j \mathrm{d}t \right),$$

où N et β sont le *lapse* et le *shift* de g respectivement. On inclut aussi dans la jauge elliptique le fait que les hypersurfaces $\{t = \textit{constante}\}$ sont maximales. Dans cette jauge, les équations d'Einstein se réécrivent comme un système couplé d'équations elliptiques semi-linéaires pour les coefficients de la métrique N, γ et β. Dans [Tou21], on démontre le théorème suivant

Théorème 1.6.4. *Si (φ, ω) sont des données initiales dans H^3 et petites dans $W^{1,4}$, alors*

1. *il existe une unique solution à (1.6.9) sur* $[0,T]\times\mathbb{R}^2$ *en jauge elliptique où* T *dépend des normes* H^3 *des données initiales,*

2. *si le temps maximal d'existence est fini, alors les normes* H^2 *de* (φ,ω) *explosent ou la petitesse dans* $W^{1,4}$ *n'est plus vérifiée.*

D'un côté, ce théorème améliore le théorème d'existence locale de [HL18a] en abaissant la régularité nécessaire pour (φ,ω) et en remplaçant la petitesse dans $W^{1,\infty}$ par la petitesse dans $W^{1,4}$. De plus, dans [HL18a] les auteurs considèrent le cas dit polarisé où $\omega=0$. D'un autre côté, on résout ici les équations d'Einstein en 3+1 dans le vide alors que dans [HL18a] les auteurs résolvent les équations d'Einstein en 3+1 couplées à une superposition de N poussières nulles. Les hypothèses et conclusions du Théorème 1.6.4 sont motivées par son application dans [HL]. En effet, le passage de $W^{1,\infty}$ à $W^{1,4}$ permet d'obtenir des estimations uniformes en N, le nombre de poussières nulles, que l'on peut alors faire tendre vers $+\infty$ pour approcher une solution du système Einstein-Vlasov.

La preuve de la première partie du Théorème 1.6.4 s'inspire de celle mise en place dans [HL18a]. Après avoir résolu les équations de contraintes, on introduit un système réduit de type transport-onde-elliptique que l'on résout en identifiant une hiérarchie d'estimées. Les équations elliptiques impliquent une meilleure régularité pour les coefficients de la métrique mais inverser le laplacien Δ sur $\mathbb{R}^2$ nous force à prendre en compte des croissances logarithmiques à l'infini.

La seconde partie du Théorème 1.6.4 montre que la norme H^2 contrôle partiellement le temps d'existence d'une solution, alors que la méthode d'énergie classique en dimension 2+1 autorise uniquement l'existence locale dans $H^{2+\epsilon}$. Pour atteindre H^2, on doit supposer que la petitesse dans $W^{1,4}$ est propagée et on met à profit une structure particulière du système application d'onde mise en évidence de manière générique dans [CB99]. En effet, on peut définir d'énergie d'ordre 3 pour les deux premières équations de (1.6.9).

Chapter 2

Solving the constraint equations

2.1 Introduction

2.1.1 Presentation of the result

In this first chapter we construct high-frequency initial data for the Einstein vacuum equations

$$R_{\mu\nu}(g) = 0 \tag{2.1.1}$$

where $R_{\mu\nu}(g)$ is the Ricci tensor of g a Lorentzian metric on the manifold $[0,1] \times \mathbb{R}^3$.

2.1.1.1 The constraint equations

Initial data for (2.1.1) on $\Sigma_0 = \{t = 0\}$ are given by

- a Riemannian metric $\bar{g}$ on Σ_0,
- a symmetric 2-tensor K on Σ_0.

In the spacetime $([0,1] \times \mathbb{R}^3, g)$ that $(\bar{g}, K)$ gives birth to, $\bar{g}$ will be the restriction of g to Σ_0 and K will be the second fundamental form of Σ_0, that is $K = -\frac{1}{2}\mathcal{L}_T g$ where T is the unit normal to Σ_0 for g. A necessary condition for $(\bar{g}, K)$ to be the set of initial data to a solution of (2.1.1) is that $(\bar{g}, K)$ solves the following *constraint equations*:

$$R(\bar{g}) + (\mathrm{tr}_{\bar{g}} K)^2 - |K|^2_{\bar{g}} = 0, \tag{2.1.2}$$

$$-\mathrm{div}_{\bar{g}} K + \mathrm{d}\mathrm{tr}_{\bar{g}} K = 0, \tag{2.1.3}$$

where $R(\bar{g})$ denotes the scalar curvature of $\bar{g}$. Equation (2.1.2) is the Hamiltonian constraint, and equation (2.1.3) is the momentum constraint. Together, they form a coupled system of non-linear elliptic partial differential equations and we refer to Chapter 7 of [CB09] for a complete presentation of their mathematical study.

2.1.1.2 High-frequency initial data and link with Chapter 3

We construct initial data adapted to the construction of Chapter 3, where we prove the existence of high-frequency solutions to (2.1.1) in generalised wave gauge. The motivation behind the study of high-frequency vacuum spacetimes is twofold: first they model rigorously the propagation of strong gravitational waves and second their behaviour in the high-frequency limit gives an example of backreaction, giving a partial answer to Burnett's conjecture (see [Bur89]).

The high-frequency solutions to (2.1.1) considered in Chapter 3 form a family $(g_\lambda)_{\lambda\in(0,1]}$ where g_λ is given by a precise high-frequency ansatz of the form

$$g_\lambda = g_0 + \lambda g^{(1)}\left(\frac{u_0}{\lambda}\right) + \lambda^2 g^{(2)}\left(\frac{u_0}{\lambda}\right) + \lambda^2\mathfrak{h}_\lambda + \lambda^3 g^{(3)}\left(\frac{u_0}{\lambda}\right) \tag{2.1.4}$$

where the $g^{(i)}$ are periodic smooth functions of their argument $\frac{u_0}{\lambda}$ and where (g_0, u_0) is a solution of the Einstein-null dust system. See Section 2.2.1 for more details on the background metric and the properties it satisfies.

Initial data for the spacetime metric g_λ given by (2.1.4) are high-frequency solutions to the constraint equations (2.1.2)-(2.1.3), i.e a couple $(\bar{g}, K)$ defined by high-frequency ansatz of the form

$$\bar{g} = \bar{g}_0 + \lambda\bar{g}^{(1)}\left(\frac{u_0}{\lambda}\right) + O\left(\lambda^2\right) \quad \text{and} \quad K = K^{(0)}\left(\frac{u_0}{\lambda}\right) + \lambda K^{(1)}\left(\frac{u_0}{\lambda}\right) + O\left(\lambda^2\right).$$

However, the high-frequency character of the spacetime we construct in Chapter 3 implies extra formal conditions that the solution of the constraint equations needs to satisfy. For instance the waves $g^{(i)}$ in (2.1.4) satisfy first order transport equations in the spacetime which schematically reads

$$\partial^\mu u_0 \mathbf{D}_\mu g^{(i)}_{\alpha\beta} = A^{(i)}_{\alpha\beta}.$$

Thus their first time derivatives are given by the RHS of these equations, while they are initially given by the second fundamental form K. Therefore, the high-frequency expansion of K is prescribed by the evolution and we need to incorporate some features of the spacetime metric g_λ in the resolution of the constraint equations. See Section 2.3.2 for a complete presentation of the interaction between the evolution and the resolution of (2.1.2)-(2.1.3).

2.1.1.3 The conformal method

In order to solve (2.1.2)-(2.1.3) we use the conformal method. This method is based on a conformal formulation of the data $(\bar{g}, K)$ and it identifies free parameters. In particular, this method transform the constraint equations (2.1.2)-(2.1.3) into a determined system of equations, a vectorial equation and a scalar equation called the Lichnerowicz equation, see Section 2.3.1.1 for a more detailed presentation.

Our main challenge is to adapt the conformal method to the present high-frequency context, which as we explained above, translates to extra formal constraints on the solution $(\bar{g}, K)$ which in turn need to be translated into constraints on the parameters of the conformal method. We refer to Section 2.3.1.2 for more details on this adaptation and in particular on the high-frequency expansions defining the parameters.

2.1.2 Preliminaries

We state our main result in Section 2.2 but before that we present the tools we will need in this chapter.

2.1.2.1 Geometric notations

Throughout this article, the notation Σ_0 refers to the manifold $\mathbb{R}^3$. On Σ_0 we consider the usual Euclidean coordinates $x = (x^1, x^2, x^3)$, and we denote by e the standard Euclidean metric, i.e

$$e = \left(\mathrm{d}x^1\right)^2 + \left(\mathrm{d}x^2\right)^2 + \left(\mathrm{d}x^3\right)^2.$$

Latin indices are used for the Euclidean coordinates and therefore runs from 1 to 3. In this chapter, repeated indices are always summed over. A second frame adapted to the background structure will be defined in Section 2.2.1.

If f is a scalar function, we define its gradient by $\nabla f = (\partial_1 f, \partial_2 f, \partial_3 f)$. If h is a Riemannian metric on Σ_0 we define

$$|\nabla f|_h^2 = h^{ij}\partial_i f \partial_j f.$$

In the particular case of the Euclidean metric we simply write $|\nabla f|$. Moreover if T and S are two symmetric 2-tensors on Σ_0, we define $|T \cdot S|_h = h^{ij}h^{k\ell}T_{ik}S_{j\ell}$ and $|T|_h^2 = |T \cdot T|_h$. The trace of T with respect to h is defined by $\mathrm{tr}_h T = h^{ij}T_{ij}$.

2.1.2.2 Function spaces and asymptotically flat manifold

If $x \in \mathbb{R}^3$ we set $\langle x \rangle := (1 + |x|)^{\frac{1}{2}}$ with $|x| = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$. We define the following weighted Sobolev spaces on $\mathbb{R}^3$.

Definition 2.1.1 (Weighted Sobolev spaces)**.** *For $1 \leq p < +\infty$, $\delta \in \mathbb{R}$ and $k \in \mathbb{N}$ we define the space $W^{k,p}_\delta$ as the completion of C^∞_c for the norm*

$$\|u\|_{W^{k,p}_\delta} = \sum_{0 \leq |\alpha| \leq k} \left\| \langle x \rangle^{\delta + |\alpha|} \nabla^\alpha u \right\|_{L^p},$$

where the L^p norm is defined with the Euclidean volume element. We extend this definition to tensors of any type by summing over all components in the Euclidean coordinates. Some special cases are $H^k_\delta := W^{k,2}_\delta$ and $L^p_\delta := W^{0,p}_\delta$.

We also define the following L^∞-based spaces:

Definition 2.1.2. *For $k \in \mathbb{N}$ and $\delta \in \mathbb{R}$ we define C^k_δ as the completion of C^∞_c for the norm*

$$\|u\|_{C^k_\delta} = \sum_{0 \leq |\alpha| \leq k} \left\| \langle x \rangle^{\delta + |\alpha|} \nabla^\alpha u \right\|_{L^\infty},$$

Let us recall some usual facts about those spaces (see [CB09] for the proofs).

Proposition 2.1.1. *Let $s, s', s_1, s_2, m \in \mathbb{N}$, $\delta, \delta', \delta_1, \delta_2, \beta \in \mathbb{R}$ and $1 \leq p < +\infty$.*

1. *If $s \leq \min(s_1, s_2)$, $s < s_1 + s_2 - \frac{3}{p}$ and $\delta < \delta_1 + \delta_2 + \frac{3}{p}$ we have the continuous embedding*

$$W^{s_1,p}_{\delta_1} \times W^{s_2,p}_{\delta_2} \subset W^{s,p}_\delta.$$

2. *If $m < s - \frac{3}{p}$ and $\beta \leq \delta + \frac{3}{p}$ we have the continuous embedding*

$$W^{s,p}_\delta \subset C^m_\beta.$$

The background metric and the solution of the constraint equations we will produce are asymptotically flat, meaning that they converge in some sense to the Euclidean metric at infinity. We give the exact definition of asymptically flat initial data.

Definition 2.1.3 (Asymptotically flat initial data)**.** *Let $\bar{g}$ be a metric on $\mathbb{R}^3$ and K a symmetric 2-tensor on $\mathbb{R}^3$. If $k > \frac{5}{2}$ and $\delta > -\frac{3}{2}$, we say that the pair $(\bar{g}, K)$ is H^k_δ asymptotically flat if*

$$\bar{g} - e \in H^k_\delta \quad \text{and} \quad K \in H^{k-1}_{\delta+1}.$$

Note that the restrictions on k and δ in the previous definition ensure that $\bar{g}$ is C^1 and $\bar{g} - e$ tends to 0 at infinity, according to Proposition 2.1.1.

2.1.2.3 Elliptic estimates

Solving the constraint equations with the conformal method requires to invert the elliptic operators Δ_γ and $\mathrm{div}_\gamma L_\gamma$, for γ a Riemannian metric. The first one is the Laplace-Beltrami operator acting on scalar functions and the second one is the conformal Laplacian acting on vector field, see Section 2.4.2 for their exact definition.

The metric γ will be defined in Section 2.4.1 but we can already say that it will be close to a background metric $\bar{g}_0$, itself close to the Euclidean metric e on $\mathbb{R}^3$. We will benefit from this fact and invert $\Delta = \Delta_e$ and $\mathrm{div}_e L_e$ rather than Δ_γ and $\mathrm{div}_\gamma L_\gamma$. The following proposition gives the desired inversion properties. The proof of its first part can be found in [McO79] while the second part is proved in [CBIY00].

Proposition 2.1.2. *If $-\frac{3}{2} < \delta < -\frac{1}{2}$ then*

$$\Delta : H^2_\delta \longrightarrow L^2_{\delta+2}$$

and

$$\mathrm{div}_e L_e : H^2_\delta \longrightarrow L^2_{\delta+2}$$

are isomorphisms.

2.1.2.4 High-frequency notations

As explained in the introduction, in this article we consider high-frequency quantities, i.e tensors of all types including scalar functions, metrics, 1-forms etc. A quantity is said to be high-frequency if it admits an expansion in powers of the small parameter λ with coefficients of the form

$$\mathrm{T}\left(\frac{u_0}{\lambda}\right) f \tag{2.1.5}$$

where f depends only on $x \in \Sigma_0$ and T is an oscillating function, i.e an element of

$$\{\theta \in \mathbb{R} \longmapsto \sin(k\theta) \mid k \in \mathbb{N}\} \cup \{\theta \in \mathbb{R} \longmapsto \cos(k\theta) \mid k \in \mathbb{N}\}. \tag{2.1.6}$$

When considering a high-frequency quantity such as a tensor S, we denote by $S^{(i)}$ the coefficients of λ^i in the expansion defining S, which thus expands formally as

$$S = \sum_{i \in \mathbb{Z}} \lambda^i S^{(i)}.$$

Note that $S^{(i)}$ is a tensor of the same type as S. Moreover, if $j \in \mathbb{Z}$ we define $S^{(\geq j)}$ by $S^{(\geq j)} = \sum_{k\geq j} \lambda^{k-j} S^{(k)}$. This allows us to clearly troncate high-frequency expansions at a fixed order as in

$$S = \sum_{k\leq j-1} \lambda^k S^{(k)} + \lambda^j S^{(\geq j)}.$$

To emphasize the fact that a high-frequency coefficient $S^{(i)}$ of a tensor S is oscillating we will often write $S^{(i)}\left(\frac{u_0}{\lambda}\right)$ instead of just $S^{(i)}$.

In order to describe concisely the *oscillating behaviour* of a high-frequency coefficient $S^{(i)}$ of a tensor S, we write for $\mathcal{A}$ a finite subset of (2.1.6)

$$S^{(i)} \overset{\text{osc}}{\sim} \sum_{\mathrm{T}\in\mathcal{A}} \mathrm{T}(\theta)$$

if there exists tensors $(S^{(i)}_{\mathrm{T}})_{\mathrm{T}\in\mathcal{A}}$ of the same type as $S^{(i)}$ such that

$$S^{(i)} = \sum_{\mathrm{T}\in\mathcal{A}} \mathrm{T}\left(\frac{u_0}{\lambda}\right) S^{(i)}_{\mathrm{T}}.$$

This notation allows us to compute the oscillating behaviour of non-linear quantities without caring too much on the non-oscillating coefficients $S^{(i)}_{\mathrm{T}}$. Note that $S^{(i)} \overset{\text{osc}}{\sim} 1$ simply means that $S^{(i)}$ is non-oscillating, i.e does not depend on $\frac{u_0}{\lambda}$.

In terms of derivation, we use the symbol θ for the derivation with respect to the $\frac{u_0}{\lambda}$ variable. For example, if f is a scalar function and if $g = \mathrm{T}\left(\frac{u_0}{\lambda}\right) f$ then

$$\partial_\theta g = \mathrm{T}'\left(\frac{u_0}{\lambda}\right) f.$$

2.2 Statement of the results

2.2.1 The background metric

In this chapter, we construct high-frequency solutions to the constraint equations which are close in some sense to a fixed background solution. Before stating our main result, we present this background solution. Denoted by $(\bar{g}_0, K_0)$, it satisfies the maximal constraint equations with sources corresponding to a null dust, that is

$$R(\bar{g}_0) - |K_0|^2_{\bar{g}_0} = 2|\nabla u_0|^2_{\bar{g}_0} F_0^2, \tag{2.2.1}$$

$$-\mathrm{div}_{\bar{g}_0} K_0 = |\nabla u_0|_{\bar{g}_0} F_0^2 \mathrm{d}u_0, \tag{2.2.2}$$

$$\mathrm{tr}_{\bar{g}_0} K_0 = 0 \tag{2.2.3}$$

where F_0 and u_0 are two scalar functions defined on $\mathbb{R}^3$. The full background solution is then $(\bar{g}_0, K_0, F_0, u_0)$ and we make some assumptions on it.

- **Assumptions on $(\bar{g}_0, K_0)$.** Even though $(\bar{g}_0, K_0)$ satisfies non-vacuum constraint equations, the sources are compactly supported (see (2.2.5) below) and we assume that $(\bar{g}_0, K_0)$ corresponds to asymptotically flat and highly regular initial data. By this we mean that there exists $\delta > -\frac{3}{2}$, a large integer N and $\varepsilon > 0$ such that

$$\|\bar{g}_0 - e\|_{H^{N+1}_\delta} + \|K_0\|_{H^N_{\delta+1}} \leq \varepsilon. \tag{2.2.4}$$

The exact threshold for N is $N \geq 10$, see Remark 2.2.4 for a justification of this particular value. For simplicity, we assume without loss of generality that $(\bar{g}_0, K_0)$ give rise to a spacetime metric such that ∂_t is the unit normal to Σ_0, which is equivalent to

$$(K_0)_{ij} = -\frac{1}{2}\partial_t(\bar{g}_0)_{ij}.$$

We denote by $\bar{\mathbf{D}}$ the covariant derivative associate to $\bar{g}_0$.

- **Assumptions on F_0.** The density F_0 is supported in a ball of size $R > 0$ in $\mathbb{R}^3$, i.e

$$\mathrm{supp}(F_0) \subset B_R := \left\{ x \in \mathbb{R}^3 \;\middle|\; |x| \leq R \right\}. \tag{2.2.5}$$

It also enjoys some regularity:

$$\|F_0\|_{H^N} \leq \varepsilon \tag{2.2.6}$$

where ε is defined above.

- **Assumptions on u_0.** There exists a constant non-zero vector field $\mathfrak{z} = (\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3)$ such that

$$\|\nabla u_0 - \mathfrak{z}\|_{H^N_{\delta+1}} \leq \varepsilon \tag{2.2.7}$$

where $\nabla u_0 = (\partial_1 u_0, \partial_2 u_0, \partial_3 u_0)$ is the euclidean gradient of u_0. By taking ε small enough in (2.2.7) we can assume that $|\nabla u_0|$ is uniformly bounded from below, which implies that u_0 has no critical points. Moreover, the level hypersurfaces of u_0 defined by

$$P_{0,u} = \left\{ x \in \mathbb{R}^3 \;\middle|\; u_0(x) = u \right\}$$

foliates $\mathbb{R}^3$ and have the topology of planes thanks to (2.2.7). This allows us to define a particular frame at each point of $\mathbb{R}^3$. We define the vector field

$$N_0 = -\frac{\bar{g}_0^{ij}\partial_i u_0 \partial_j}{|\nabla u_0|_{\bar{g}_0}}.$$

It satisfies $\bar{g}_0(N_0, N_0) = 1$ and is orthogonal to the hypersurface $P_{0,u}$. We consider at each point $x \in \mathbb{R}^3$ an orthonormal basis $(e_{\mathbf{1}}, e_{\mathbf{2}})$ of $T_x P_{0,u}$ for the metric $\bar{g}_0$. The frame $(N_0, e_{\mathbf{1}}, e_{\mathbf{2}})$ will play a crucial role in our construction. While we reserve the usual latin indices for the coordinates system (x^1, x^2, x^3), i.e $i, j \in \{1, 2, 3\}$, the bold latin indices are used for the frame $(e_{\mathbf{1}}, e_{\mathbf{2}})$, i.e $\mathbf{i}, \mathbf{j} \in \{\mathbf{1}, \mathbf{2}\}$.

Finally, we assume that u_0 give rise to a spacetime solution of the eikonal equation, thus we assume that $\partial_t u_0 = |\nabla u_0|_{\bar{g}_0}$.

In this article, we don't prove that a background solution $(\bar{g}_0, K_0, F_0, u_0)$ solving (2.2.1)-(2.2.3) and satisfying the above assumptions exists. We refer again to [CBIY00] for the details of how one can solve the constraint equations with sources in the asymptotically flat setting.

Remark 2.2.1. *As explained in the introduction, the main goal of this article is to construct initial data for Chapter 3. The background solution $(\bar{g}_0, K_0, F_0, u_0)$ considered here is actually the initial data of the background* spacetime *solution considered in Chapter 3, which solves the Einstein null dust system. In the same spirit, the frame $(N_0, e_{\mathbf{1}}, e_{\mathbf{2}})$ is the projection of the background null frame $(L_0, \underline{L}_0, e_{\mathbf{1}}, e_{\mathbf{2}})$ considered in Chapter 3.*

Remark 2.2.2. *The fact that the background solves the maximal constraint equations and not the full constraint equations is not an important requirement, and our construction can be adapted to the case where* $(\bar{g}_0, K_0, F_0, u_0)$ *solves*

$$R(\bar{g}_0) + (\mathrm{tr}_{\bar{g}_0} K_0)^2 - |K_0|^2_{\bar{g}_0} = 2|\nabla u_0|^2_{\bar{g}_0} F_0^2,$$
$$-\mathrm{div}_{\bar{g}_0} K_0 + \mathrm{tr}_{\bar{g}_0} K_0 = |\nabla u_0|_{\bar{g}_0} F_0^2 \mathrm{d}u_0.$$

However, we choose to include the relation (2.2.3) since it simplifies our computations.

2.2.2 Solving the constraint equations

The following theorem is the main result of this chapter.

Theorem 2.2.1. *Let* $(\bar{g}_0, K_0, F_0, u_0)$ *be the solution of the maximal constraint equations coupled with a null dust described in Section 2.2.1, and let* $\varepsilon > 0$ *be the smallness threshold. There exists* $\varepsilon_0 = \varepsilon_0(\delta, R) > 0$ *such that if* $0 < \varepsilon \leq \varepsilon_0$*, there exists for all* $\lambda \in (0, 1]$ *a solution* $(\bar{g}_\lambda, K_\lambda)$ *solution of the constraint equations (3.1.14)-(3.1.15) on* $\mathbb{R}^3$ *of the form*

$$\bar{g}_\lambda = \bar{g}_0 + \lambda \cos\left(\frac{u_0}{\lambda}\right) \bar{F}^{(1)} + \lambda^2 \left(\sin\left(\frac{u_0}{\lambda}\right) \bar{F}^{(2,1)} + \cos\left(\frac{2u_0}{\lambda}\right) \bar{F}^{(2,2)} \right) + \lambda^2 \bar{\mathfrak{h}}_\lambda, \tag{2.2.8}$$

$$K_\lambda = K_\lambda^{(0)} + \lambda K_\lambda^{(1)} + \lambda^2 K_\lambda^{(\geq 2)}, \tag{2.2.9}$$

with

$$K_\lambda^{(0)} = K_0 + \frac{1}{2} \sin\left(\frac{u_0}{\lambda}\right) |\nabla u_0|_{\bar{g}_0} \bar{F}^{(1)}, \tag{2.2.10}$$

$$\begin{aligned}\left(K_\lambda^{(1)}\right)_{ij} = -\frac{1}{2} \cos\left(\frac{u_0}{\lambda}\right) \left(-N_0 \bar{F}^{(1)}_{ij} + (\partial_t + N_0)^\rho \Gamma(g_0)^k_{\rho(i} \bar{F}^{(1)}_{j)k} + \frac{1}{2|\nabla u_0|_{\bar{g}_0}} (\Box_{g_0} u_0) \bar{F}^{(1)}_{ij} \right) \\ - \frac{1}{2} |\nabla u_0|_{\bar{g}_0} \left(\cos\left(\frac{u_0}{\lambda}\right) \bar{F}^{(2,1)}_{ij} - 2 \sin\left(\frac{2u_0}{\lambda}\right) \bar{F}^{(2,2)}_{ij} \right).\end{aligned} \tag{2.2.11}$$

Moreover:

(i) the tensors $\bar{F}^{(1)}$, $\bar{F}^{(2,1)}$ *and* $\bar{F}^{(2,2)}$ *are supported in* $\{|x| \leq R\}$ *and there exists* $C_{\mathrm{cons}} = C_{\mathrm{cons}}(\delta, R) > 0$ *such that*

$$\left\| \bar{F}^{(1)} \right\|_{H^N} + \left\| \bar{F}^{(2,1)} \right\|_{H^{N-1}} + \left\| \bar{F}^{(2,2)} \right\|_{H^{N-1}} \leq C_{\mathrm{cons}} \varepsilon,$$

(ii) the tensor $\bar{F}^{(1)}$ *is* $\bar{g}_0$*-traceless, tangential to* $P_{0,u}$ *and satisfies*

$$\left| \bar{F}^{(1)} \right|^2_{\bar{g}_0} = 8 F_0^2, \tag{2.2.12}$$

(iii) the tensors $\bar{\mathfrak{h}}_\lambda$ *and* $K_\lambda^{(\geq 2)}$ *belong to the spaces* H^5_δ *and* $H^4_{\delta+1}$ *respectively and satisfy*

$$\max_{r \in [\![0,4]\!]} \lambda^r \left\| \nabla^{r+1} \bar{\mathfrak{h}}_\lambda \right\|_{L^2_{\delta+r+1}} \leq C_{\mathrm{cons}} \varepsilon, \tag{2.2.13}$$

$$\max_{r \in [\![0,4]\!]} \lambda^r \left\| \nabla^r K_\lambda^{(\geq 2)} \right\|_{L^2_{\delta+r+1}} \leq C_{\mathrm{cons}} \varepsilon. \tag{2.2.14}$$

Remark 2.2.3. *Note that in Theorem 2.2.1, the only free data are the background quantities* $(\bar{g}_0, K_0, F_0, u_0)$. *The tensors* $\bar{F}^{(1)}$, $\bar{F}^{(2,1)}$ *and* $\bar{F}^{(2,2)}$ *are obtained in the proof and are determined by* $(\bar{g}_0, K_0, F_0, u_0)$. *The only place where we have freedom in the choice of these tensors is when choosing the polarization of* $\bar{F}^{(1)}$, *see the definition of* $\omega^{(1)}$ *in Section 2.4.1.*

Remark 2.2.4. *The threshold* $N \geq 10$ *is chosen so that all the oscillating quantities in this chapter can be estimated in* L^∞ *along with enough derivatives so that (2.2.13) and (2.2.14) hold. If* N *is larger, then* $\mathfrak{h}_\lambda$ *or* $K_\lambda^{(\geq 2)}$ *would be more regular than what is stated in Theorem 2.2.1 but we choose only to state the minimal regularity required by our application in Chapter 3.*

Remark 2.2.5. *In (2.2.11), the index* ρ *in* $(\partial_t + N_0)^\rho \Gamma(g_0)^k_{\rho i}$ *is a spacetime index and runs from 0 to 3. Therefore this term seems to depend on the spacetime metric* g_0 *and not only on the solution of the equations (2.2.1)-(2.2.3). However, since we assume that* $(\bar{g}_0, K_0)$ *give rise to initial data on* Σ_0 *for (2.1.1) such that* ∂_t *is the unit normal to* Σ_0, *we have* $\Gamma(g_0)^k_{0i} = -\bar{g}_0^{k\ell}(K_0)_{i\ell}$. *A similar argument can be made for the term* $\Box_{g_0} u_0$, *which can be expressed only with* $(\bar{g}_0, K_0)$ *and spatial derivatives of* u_0 *using in particular* $\partial_t u_0 = |\nabla u_0|_{\bar{g}_0}$. *The expression of* $K_\lambda^{(1)}$ *and its link with the spacetime metric will be motivated in Section 2.3.2.2.*

As explained above, the main motivation behind Theorem 2.2.1 is to construct high-frequency initial data for the Einstein vacuum equations, which are solved in Chapter 3. However, to the best of the author's knowledge, Theorem 2.2.1 also constitutes the first example of backreaction for the constraint equations of general relativity. Indeed, the sequence $(\bar{g}_\lambda, K_\lambda)_\lambda$ solves the *vacuum* constraint equations while converging in the following sense

$$
\begin{aligned}
\bar{g}_\lambda \to \bar{g}_0, &\quad \text{uniformly in } L^\infty, \\
\nabla \bar{g}_\lambda \rightharpoonup \nabla \bar{g}_0, &\quad \text{weakly in } L^2_{loc}, \\
K_\lambda \rightharpoonup K_0, &\quad \text{weakly in } L^2_{loc},
\end{aligned}
$$

towards a solution of the constraint equations *with sources* (see (2.2.1)-(2.2.2)-(2.2.3)).

From now on and until Section 2.7 we drop the index λ in the solution $(\bar{g}_\lambda, K_\lambda)$ obtained in Theorem 2.2.1 and simply write $(\bar{g}, K)$.

2.3 Strategy of proof

In this section, we present the strategy of the proof of Theorem 2.2.1. Our main tool is a high-frequency adaptation of the conformal method which we present in Section 2.3.1. In Section 2.3.2 we present our main formal challenge, that is the compatibility with the spacetime ansatz.

2.3.1 High-frequency conformal method

We start by recalling what is the conformal method.

2.3.1.1 The conformal method

Introduced by Lichnerowicz in [Lic44], this method aims at solving the constraint equations (2.1.2) and (2.1.3) by requiring the solution $(\bar{g}, K)$ to satisfy formal geometric properties.

The idea giving its name to the method is to prescribe the conformal class of $\bar{g}$, i.e to fix a Riemannian metric γ on $\mathbb{R}^3$ and to solve for the scalar function φ such that

$$\bar{g} = \varphi^4 \gamma. \tag{2.3.1}$$

We say that a tensor is a TT-tensor if it is traceless divergence free and symmetric and one can show that the space of TT-tensor depends only on the conformal class of the metric. Therefore, the next step in the conformal method is to decompose the symmetric 2-tensor K in connection with its trace and divergence features. More precisely we fix a scalar function τ and a TT-tensor σ for the metric γ and solve for the vector field W such that

$$K = \varphi^{-2}(\sigma + L_\gamma W) + \frac{1}{3}\varphi^4 \gamma \tau \tag{2.3.2}$$

where $L_\gamma W$ is defined in (2.4.30). The exponents appearing in (2.3.1) and (2.3.2) are linked to the dimension of the manifold, here 3, see Chapter 6 of [CB09] for their general expression. The constraint equations (2.1.2) and (2.1.3) now rewrite as the following coupled system of non-linear elliptic equations for (φ, W):

$$8\Delta_\gamma \varphi = R(\gamma)\varphi + \frac{2}{3}\tau^2\varphi^5 - |\sigma + L_\gamma W|^2_\gamma \varphi^{-7}, \tag{2.3.3}$$

$$\mathrm{div}_\gamma L_\gamma W = \frac{2}{3}\varphi^6 \mathrm{d}\tau. \tag{2.3.4}$$

2.3.1.2 High-frequency expansions of the parameters and the solutions

In this chapter, we want to construct high-frequency solutions to the constraint equations through the conformal method. This requires the parameters of the conformal method (that is γ, τ and σ) as well as the solutions φ and W to be defined by high-frequency expansions. Let us detail their definitions.

We first consider the high-frequency metric γ. Since we want $\bar{g}$ to be close to $\bar{g}_0$, we choose γ already close to $\bar{g}_0$. More precisely, the full ansatz for γ is

$$\gamma = \bar{g}_0 + \lambda\gamma^{(1)}\left(\frac{u_0}{\lambda}\right) + \lambda^2\gamma^{(2)}\left(\frac{u_0}{\lambda}\right), \tag{2.3.5}$$

where we recall that the notation $\gamma^{(i)}\left(\frac{u_0}{\lambda}\right)$ is used to emphasize the fact that $\gamma^{(i)}$ is a linear combination of terms of the form (2.1.5).

Similarly, since we want K to be close to K_0 (in a weak sense since it is at the level of one derivative of the metric, see the discussion following Remark 2.2.5) and since we assume that K_0 is $\bar{g}_0$ traceless, we define τ to be of order λ^1, i.e

$$\tau = \lambda\tau^{(1)}\left(\frac{u_0}{\lambda}\right). \tag{2.3.6}$$

The fact that the high-frequency perturbations don't contribute to the mean curvature at the λ^0 order will be justified in the proof. The TT-tensor σ requires a special construction linked to what we call the compatibility with the spacetime ansatz (see Section 2.3.2.2), but we can already give its high-frequency expansion:

$$\sigma = \sigma^{(0)}\left(\frac{u_0}{\lambda}\right) + \lambda\sigma^{(1)}\left(\frac{u_0}{\lambda}\right) + \lambda^2\left(\sigma^{(2)}\left(\frac{u_0}{\lambda}\right) + \widetilde{\sigma}\right) \tag{2.3.7}$$

where $\widetilde{\sigma}$ is a remainder whose role is to solve the equations defining the class of TT-tensors, that is $\mathrm{div}_\gamma\sigma = 0$ and $\mathrm{tr}_\gamma\sigma = 0$, with γ defined by (2.3.5).

To understand the expansions for the solutions φ and W, let us look at the effect of the expansions (2.3.5)-(2.3.6)-(2.3.7) on the equations (2.3.3)-(2.3.4). A standard feature of high-frequency quantities is that they lose one power of λ per derivative. In (2.3.3) the metric γ is only differentiated in the scalar curvature $R(\gamma)$ and we can derive from (2.3.5) that

$$R(\gamma) = \frac{|\nabla u_0|^2_{\bar{g}_0}}{\lambda} \partial_\theta^2 \left(\gamma^{(1)}_{N_0 N_0} - \mathrm{tr}_{\bar{g}_0} \gamma^{(1)}_{ij} \right) + O\left(\lambda^0\right). \tag{2.3.8}$$

Therefore, if $\gamma^{(1)}$ is well-chosen, the RHS of (2.3.3) is $O\left(\lambda^0\right)$. This implies that the first oscillating term in φ must be at order λ^2, since Δ_γ is a second order elliptic operator (see also Remark 2.3.1). More precisely we choose

$$\varphi = 1 + \lambda^2 \left(\varphi^{(2)} \left(\frac{u_0}{\lambda} \right) + \widetilde{\varphi} \right) + \lambda^3 \varphi^{(3)} \left(\frac{u_0}{\lambda} \right) \tag{2.3.9}$$

where $\widetilde{\varphi}$ is a non-oscillating remainder. The constant coefficient in (2.3.9) ensures that $\bar{g}_\lambda = \bar{g}_0 + O(\lambda)$. Similarly, (2.3.6) and (2.3.9) imply that $\varphi^6 \mathrm{d}\tau = O\left(\lambda^0\right)$ and the ellipticity of $\mathrm{div}_\gamma L_\gamma$ allows us to choose W of the following form

$$W = \lambda^2 \left(W^{(2)} \left(\frac{u_0}{\lambda} \right) + \widetilde{W} \right) + \lambda^3 W^{(3)} \left(\frac{u_0}{\lambda} \right) \tag{2.3.10}$$

Remark 2.3.1. *The ellipticity of Δ_γ, or the fact that γ is a Riemannian metric, plays a crucial role here. Indeed, we have*

$$\Delta_\gamma \left(f \left(\frac{u_0}{\lambda} \right) \right) = \frac{|\nabla u_0|^2_{\bar{g}_0}}{\lambda^2} \partial_\theta^2 f + O\left(\frac{1}{\lambda} \right)$$

whereas if g is a Lorentzian metric, the action of the wave operator on oscillating function is given by

$$\Box_g \left(f \left(\frac{u_0}{\lambda} \right) \right) = \frac{1}{\lambda} \partial_\theta \left(2\partial^\mu u_0 \partial_\mu f + (\Box_g u_0) f \right) + O\left(\lambda^0\right) \tag{2.3.11}$$

where u_0 satisfies in this case the eikonal equation (this is the standard geometric optics approach for hyperbolic equations). This means that in the Lorentzian case, a λ^0 order term is absorbed by adding a λ^1 wave whereas in the Riemannian case a λ^2 wave would be enough. A similar remark applies for the conformal Laplacian $\mathrm{div}_\gamma L_\gamma$.

As is standard when considering high-frequency expansions, we obtain a hierarchy of equations. Very schematically, one can say that $\varphi^{(2)}$ and $\varphi^{(3)}$ (resp. $W^{(2)}$ and $W^{(3)}$) will solve the orders λ^0 and λ^1 of (2.3.3) (resp. (2.3.4)), while the non-oscillating remainder $\widetilde{\varphi}$ (resp. $\widetilde{W}$) will solve the orders higher than λ^2. Compared to a standard high-frequency expansions, the difficulty here is that the coefficients of the equations themselves, i.e the parameters of the conformal method, are also defined through high-frequency expansions and must satisfy formal constraints, the main one being that σ must be a TT-tensor. Therefore, our construction is intricate and we present now some aspects of this entanglement.

2.3.2 Formal constraints

The main purpose of the sequence $(\bar{g}_\lambda, K_\lambda)_\lambda$ is to define special initial data for a family of spacetime metrics solving the Einstein vacuum equations, i.e a family of Ricci flat Lorentzian metrics. These metrics g_λ are also defined by a high-frequency expansion of the form

$$g_\lambda = g_0 + \lambda g^{(1)} \left(\frac{u_0}{\lambda} \right) + \lambda^2 g^{(2)} \left(\frac{u_0}{\lambda} \right) + \lambda^2 \tilde{\mathfrak{h}}_\lambda$$

where g_0 is the spacetime background metric, see Theorem 3.1.2 in Chapter 3. The high-frequency character of the vacuum metric g_λ has several formal consequences and it imposes extra constraints on the solution $(\bar{g}_\lambda, K_\lambda)$ of the constraint equations, since we want $(\bar{g}_\lambda, K_\lambda)$ to be the first and second fundamental form of the hypersurface Σ_0 in the spacetime $([0,1]\times\mathbb{R}^3, g_\lambda)$ constructed in Chapter 3.

2.3.2.1 The first fundamental form

As explained in Section 3.2.1.2 of Chapter 3, the waves $g^{(1)}$ and $g^{(2)}$ need to satisfy algebraic *polarization* conditions, which concretely prescribe their polarization tensor

$$\mathrm{Pol}_\alpha\left(g^{(i)}\right) = g_0^{\mu\nu}\left(\partial_\mu u_0 g^{(i)}_{\nu\alpha} - \frac{1}{2}\partial_\alpha u_0 g^{(i)}_{\mu\nu}\right).$$

These polarization conditions must hold initially on Σ_0 which *a priori* imposes extra constraints on $\bar{g}$. While the polarization conditions for $g^{(2)}$ are ensured by the choice of the normal components and hence are independent from $\bar{g}$ (see Section 3.4.6.1 in Chapter 3), the polarization conditions for $g^{(1)}$ are ensured by $\bar{g}$. Since the latter is given by (2.3.1) and since $\varphi = 1 + O\left(\lambda^2\right)$ (see (2.3.9)), $g^{(1)}_{ij}$ is actually given by $\gamma^{(1)}$. Therefore, the polarization condition for $g^{(1)}$ is initially ensured through a well-chosen parameter $\gamma^{(1)}$. The exact condition that $\gamma^{(1)}$ needs to satisfy is to be $P_{0,u}$-tangential and $\bar{g}_0$-traceless, see (2.4.4) and (2.4.5).

Remark 2.3.2. *These conditions correspond exactly to the definition of the TT gauge in the linearized gravity setting for a plane wave propagating in the N_0 direction. Note that N_0 is not a constant vector field, therefore strictly speaking a plane wave can't propagate in the N_0 direction. However, as (2.2.7) shows, N_0 is close to the $\mathfrak{z}$ direction and the analogy with the TT gauge of linearized gravity is thus valid.*

Remark 2.3.3. *Actually, the polarization condition for $g^{(1)}_{ij} = \gamma^{(1)}_{ij}$ precisely implies $R(\gamma) = O\left(\lambda^0\right)$. Indeed, recalling (2.3.8), if $\gamma^{(1)}$ is $P_{0,u}$-tangential and $\bar{g}_0$-traceless then*

$$\gamma^{(1)}_{N_0N_0} = \mathrm{tr}_{\bar{g}_0}\gamma^{(1)} = 0$$

and we get $R(\gamma) = O\left(\lambda^0\right)$. As explained above, the expansion $\varphi = 1 + O\left(\lambda^2\right)$ is made possible by $R(\gamma) = O\left(\lambda^0\right)$.

Another formal constraint on the spacetime metric g_λ is linked to the backreaction at the heart of Burnett's conjecture. As explained in Chapter 3, the background metric g_0 solves the Einstein equations with sources which are produced by the self-interaction of the wave $g^{(1)}$ through the quadratic non-linearity of the Ricci tensor. The non-triviality of our construction (i.e the fact that g_0 does not describe a vacuum spacetime) is thus ensured by some quadratic contractions of the tensor $g^{(1)}$. This translates as the condition (2.4.6) on $\gamma^{(1)}$.

2.3.2.2 The second fundamental form and the TT-tensor

As (2.3.11) shows, the waves $g^{(1)}$ and $g^{(2)}$ will solve *first order* transport equations along the rays of the phase u_0 (recall that in generalised wave gauge the principal part of the Ricci tensor is a wave operator). For such equations, the Cauchy problem only requires that the initial value of the solution is given on Σ_0, while the normal derivative is given by the equation itself. This implies that $\partial_t g^{(1)}_{ij} \restriction \Sigma_0$ is actually prescribed by the evolution equations, but it is also given by the second fundamental form of Σ_0, i.e K_λ. Therefore, there is a formal constraint coming from the evolution on the 2-tensor K_λ obtained by solving the constraint equations. We say that K_λ must satisfy the *compatibility with the spacetime ansatz.*

Let us be more precise on this formal constraint. If $\lambda g_{ij}^{(1)}$ is of the form $\lambda \cos\left(\frac{u_0}{\lambda}\right) F_{ij}^{(1)}$ with $F_{ij}^{(1)}$ solving a transport equation, then its time derivative is of the form

$$- \sin\left(\frac{u_0}{\lambda}\right) |\nabla u_0|_{\bar{g}_0} F_{ij}^{(1)} + \lambda \cos\left(\frac{u_0}{\lambda}\right) \partial_t F_{ij}^{(1)} \tag{2.3.12}$$

where we used the fact that $\partial_t u_0 = |\nabla u_0|_{\bar{g}_0}$ on Σ_0, as u_0 solves the background eikonal equation in the full spacetime. The λ^0 part of (2.3.12) contributes to $K_\lambda^{(0)}$ (see (2.2.10)) while the λ^1 part contributes to $K_\lambda^{(1)}$ (see the first line of (2.2.11)). In (2.2.11), the contribution of $\partial_t F_{ij}^{(1)}$ has been replaced by its value given by the transport equation that $F^{(1)}$ satisfies in the spacetime, i.e

$$-2\mathbf{D}_{L_0} F^{(1)} + (\Box_{g_0} u_0) F^{(1)} = 0, \tag{2.3.13}$$

so that we obtain an expression of $K_\lambda^{(1)}$ involving only $\bar{g}$, i.e the induced metric on Σ_0 ($L_0 = -g_0^{\alpha\beta}\partial_\alpha u_0 \partial_\beta$ in (2.3.13)).

The second fundamental form is defined by $-\frac{1}{2}\mathcal{L}_T g_\lambda$ where $\mathcal{L}$ denotes the Lie derivative and T is the unit normal to Σ_0 for g_λ. Therefore, incorporating these formal constraints in the expression of $K_\lambda^{(0)}$ and $K_\lambda^{(1)}$ require to have the expression of T, which *a priori* depends on λ. However, as it is shown in Section 3.1.3 of Chapter 3, we can choose the normal components of the initial data such that $T = \partial_t + O\left(\lambda^2\right)$. This fully justifies the ansatz for K_λ obtained in Theorem 2.2.1.

This interaction between the evolution and the solution of the contraint equations has a consequence for the choice of the TT-tensor σ. Indeed, in the usual way of solving the constraint equations, one starts by choosing σ and then obtains K through the formula (2.3.2). Here, as we explained, the orders λ^0 and λ^1 of K are prescribed by the evolution. Since $\varphi = 1 + O\left(\lambda^2\right)$ (which implies that φ plays no role in the order λ^0 and λ^1 in (2.3.2)), this defines the first orders of σ, i.e $\sigma^{(0)}$ and $\sigma^{(1)}$ in (2.3.7) so that (2.3.2) holds. We are then left with the task of constructing a TT-tensor with prescribed first orders in its high-frequency expansion. This is the content of Sections 2.5.3 and 2.5.4.

2.3.3 Outline of the proof

We give here the outline of the rest of this chapter, which proves Theorem 2.2.1.

- In Section 2.4 we define the metric γ and compute useful high-frequency expansions such as its scalar curvature $R(\gamma)$ or differential operators depending on γ and appearing in the constraint equations.
- In Section 2.5 we define $\varphi^{(2)}$, $\varphi^{(3)}$ and $W^{(2)}$ such that they solve the first orders of the constraint equations and by doing so we also define the parameter τ. This section is concluded by the construction of the TT-tensor σ.
- In Section 2.6 we fully solve the constraint equations with a fixed point argument for the remainders $\widetilde{\varphi}$ and $\widetilde{W}$ (we also define $W^{(3)}$ in the process).
- In Section 2.7 we conclude the proof of Theorem 2.2.1.

2.4 High-frequency conformal class

In this section we define the metric γ on Σ_0, that is the prefered member of the conformal class in which we look for $\bar{g}$ according to (2.3.1).

2.4.1 Definitions and first computations

We choose

$$\gamma = \bar{g}_0 + \lambda\gamma^{(1)} + \lambda^2\gamma^{(2)} \tag{2.4.1}$$

where

$$\gamma^{(1)} = \cos\left(\frac{u_0}{\lambda}\right) F_0\omega^{(1)}, \tag{2.4.2}$$

$$\gamma^{(2)} = \sin\left(\frac{u_0}{\lambda}\right) \omega^{(2)}. \tag{2.4.3}$$

The two symmetric 2-forms $\omega^{(1)}$ and $\omega^{(2)}$ are directly defined in the frame $(N_0, e_{\mathbf{1}}, e_{\mathbf{2}})$.

- **Definition of $\omega^{(1)}$.** The coefficients of $\omega^{(1)}$ in the frame $(N_0, e_{\mathbf{1}}, e_{\mathbf{2}})$ are constants and satisfy

$$\omega^{(1)}_{N_0 i} = 0, \tag{2.4.4}$$

$$\omega^{(1)}_{\mathbf{11}} + \omega^{(1)}_{\mathbf{22}} = 0, \tag{2.4.5}$$

$$\left(\omega^{(1)}_{\mathbf{11}}\right)^2 + \left(\omega^{(1)}_{\mathbf{12}}\right)^2 = 4. \tag{2.4.6}$$

- **Definition of $\omega^{(2)}$.** The 2-form $\omega^{(2)}$ depends on $\omega^{(1)}$ in the following way: all the coefficients of $\omega^{(2)}$ in the frame $(N_0, e_{\mathbf{1}}, e_{\mathbf{2}})$ are set to be zero except $\omega^{(2)}_{N_0\mathbf{1}}$ and $\omega^{(2)}_{N_0\mathbf{2}}$ which are defined by

$$\omega^{(2)}_{N_0\mathbf{j}} = \frac{1}{|\nabla u_0|^2_{\bar{g}_0}}(\mathrm{div}_{\bar{g}_0}|\nabla u_0|_{\bar{g}_0}F_0\omega^{(1)})_{\mathbf{j}} - \frac{1}{|\nabla u_0|_{\bar{g}_0}}F_0\omega^{(1)}_{i\mathbf{j}}\left(\mathbf{D}_{N_0}N_0^i - \bar{g}_0^{ik}(K_0)_{N_0 k}\right) \tag{2.4.7}$$

  This expression will be justified in the proof of Lemma 2.5.3, where we construct the TT-tensor of the conformal method.

Remark 2.4.1. *The properties* (2.4.4) *and* (2.4.5) *and the symmetry of* $\omega^{(1)}$ *imply that the tensor* $\gamma^{(1)}$ *is a linear combination of the tensors*

$$e_{\mathbf{1}} \otimes e_{\mathbf{1}} - e_{\mathbf{2}} \otimes e_{\mathbf{2}} \quad \textit{and} \quad e_{\mathbf{1}} \otimes e_{\mathbf{2}} + e_{\mathbf{2}} \otimes e_{\mathbf{1}}.$$

This is the equivalent of the 2 degrees of freedom or polarization of the TT gauge in the linearized gravity setting where g_0 *is replaced by the Minkowskian metric, see Remark 2.3.2. As noted in Remark 2.2.3, the choice of the polarization of* $\omega^{(1)}$*, i.e the choice of* $\omega^{(1)}_{\mathbf{11}}$ *and* $\omega^{(1)}_{\mathbf{12}}$ *under the constraint* (2.4.6)*, is the only freedom we have in the definition of the tensors* $\bar{F}^{(1)}$*,* $\bar{F}^{(2,1)}$ *and* $\bar{F}^{(2,2)}$*.*

We define

$$\bar{F}^{(1)} = F_0\omega^{(1)}. \tag{2.4.8}$$

The following lemma summarizes the important properties of $\bar{F}^{(1)}$.

Lemma 2.4.1. *The symmetric 2-tensor $\bar{F}^{(1)}$ is supported in B_R, $\bar{g}_0$-traceless and $P_{0,u}$-tangential. Moreover, (2.2.12) holds and*

$$\left\| \bar{F}^{(1)} \right\|_{H^N} \lesssim \varepsilon \tag{2.4.9}$$

with a constant depending only on δ and R.

Proof. The support property is implied by the support property of F_0. The property (2.4.4) implies that $\bar{F}^{(1)}$ is $P_{0,u}$-tangential, which together with (2.4.5) implies that $\bar{F}^{(1)}$ is $\bar{g}_0$-traceless since

$$\mathrm{tr}_{\bar{g}_0} \bar{F}^{(1)} = \bar{F}^{(1)}_{N_0 N_0} + \bar{F}^{(1)}_{\mathbf{11}} + \bar{F}^{(1)}_{\mathbf{22}}.$$

Moreover, (2.4.6) and (2.4.8) imply (2.2.12). The estimation (2.4.9) comes from (2.2.6) and (2.2.7), the latter allowing us to estimate the coefficients of $\omega^{(1)}$ in the usual Euclidean coordinates, i.e $\omega^{(1)}_{ij}$. □

The following lemma gives the expansion of the scalar curvature of γ. This will be used to solve the Hamitonian constraint (2.3.3).

Lemma 2.4.2. *We have*

$$R(\gamma) = R^{(0)} + \lambda R^{(1)} + \lambda^2 R^{(\geq 2)}$$

with

$$\begin{aligned} R^{(0)} &= R(\bar{g}_0) - |\nabla u_0|^2_{\bar{g}_0} F_0^2 - 7\cos\left(\frac{2u_0}{\lambda}\right) |\nabla u_0|^2_{\bar{g}_0} F_0^2 \\ &\quad + \sin\left(\frac{u_0}{\lambda}\right) |\nabla u_0|_{\bar{g}_0} \bar{F}^{(1)}_{\ell j} \left(-\bar{g}_0^{ij} \partial_i N_0^\ell + \frac{1}{2} N_0 \bar{g}_0^{\ell j} \right) \end{aligned} \tag{2.4.10}$$

and

$$\left| R^{(1)} \right| + \left| R^{(\geq 2)} \right| \lesssim \left| \gamma^{-2} \partial^2 \gamma \right| + \left| \gamma^{-3} (\partial\gamma)^2 \right|. \tag{2.4.11}$$

Moreover

$$R^{(1)} \overset{\text{osc}}{\sim} \cos(\theta) + \sin(2\theta) + \cos(3\theta). \tag{2.4.12}$$

Proof. We recall the definition of the scalar curvature

$$R(\gamma) = \gamma^{ij} \left(\partial_k \Gamma(\gamma)^k_{ij} - \partial_i \Gamma(\gamma)^k_{jk} + \Gamma(\gamma)^k_{k\ell} \Gamma(\gamma)^\ell_{ij} - \Gamma(\gamma)^k_{i\ell} \Gamma(\gamma)^\ell_{jk} \right)$$

We start by giving an estimation up to second order in λ of the inverse of γ:

$$\begin{aligned} \gamma^{ij} &= \bar{g}_0^{ij} - \lambda \cos\left(\frac{u_0}{\lambda}\right) (\bar{F}^{(1)})^{ij} \\ &\quad + \lambda^2 \left(\cos^2\left(\frac{u_0}{\lambda}\right) \bar{g}_0^{ik} (\bar{F}^{(1)})^{j\ell} \bar{F}^{(1)}_{k\ell} - \sin\left(\frac{u_0}{\lambda}\right) (\omega^{(2)})^{ij} \right) + O\left(\lambda^3\right) \end{aligned} \tag{2.4.13}$$

where on the RHS all the inverses are taken with respect to the background metric $\bar{g}_0$. We now expand the Christoffel symbols of γ, we obtain

$$\Gamma^k_{ij} = \Gamma(\bar{g}_0)^k_{ij} + (\tilde{\Gamma}^{(0)})^k_{ij} + \lambda (\Gamma^{(1)})^k_{ij} + O\left(\lambda^2\right) \tag{2.4.14}$$

with

$$(\tilde{\Gamma}^{(0)})^k_{ij} = -\frac{1}{2}\sin\left(\frac{u_0}{\lambda}\right)\bar{g}_0^{k\ell}\left(\partial_{(i}u_0\bar{F}^{(1)}_{\ell j)} - \partial_\ell u_0 \bar{F}^{(1)}_{ij}\right) \tag{2.4.15}$$

$$\begin{aligned}(\Gamma^{(1)})^k_{ij} &= \frac{1}{2}\cos\left(\frac{u_0}{\lambda}\right)\bar{g}_0^{k\ell}\left(\partial_{(i}u_0\omega^{(2)}_{\ell j)} - \partial_\ell u_0 \omega^{(2)}_{ij}\right) \\ &\quad + \frac{1}{4}\sin\left(\frac{2u_0}{\lambda}\right)(\bar{F}^{(1)})^{k\ell}\partial_{(i}u_0\bar{F}^{(1)}_{\ell j)} + \cos\left(\frac{u_0}{\lambda}\right)\bar{Q}^k_{ij}\end{aligned} \tag{2.4.16}$$

where we defined

$$\bar{Q}^k_{ij} = \frac{1}{2}\bar{g}_0^{k\ell}\left(\partial_{(i}\bar{F}^{(1)}_{\ell j)} - \partial_\ell \bar{F}^{(1)}_{ij}\right) - \frac{1}{2}(\bar{F}^{(1)})^{k\ell}\left(\partial_{(i}(\bar{g}_0)_{\ell j)} - \partial_\ell(\bar{g}_0)_{ij}\right).$$

By using $\bar{F}^{(1)}_{N_0 i} = 0$ and $\mathrm{tr}_{\bar{g}_0}\bar{F}^{(1)} = 0$, we can compute useful contractions of $\bar{Q}^k_{ij}$. We first look at $\bar{Q}^k_{jk}$ which vanishes thanks to $\mathrm{tr}_{\bar{g}_0}\bar{F}^{(1)} = 0$:

$$\bar{Q}^k_{jk} = \frac{1}{2}\left(\bar{g}_0^{k\ell}\partial_j\bar{F}^{(1)}_{\ell k} - (\bar{F}^{(1)})^{k\ell}\partial_j(\bar{g}_0)_{\ell k}\right) = \frac{1}{2}\partial_j \mathrm{tr}_{\bar{g}_0}\bar{F}^{(1)} = 0 \tag{2.4.17}$$

Using in addition $\bar{F}^{(1)}_{N_0 i} = 0$ we have:

$$\begin{aligned}\bar{g}_0^{ij}(N_0)_k\bar{Q}^k_{ij} &= \bar{g}_0^{ij}N_0^\ell\left(\partial_i\bar{F}^{(1)}_{\ell j} - \frac{1}{2}\partial_\ell\bar{F}^{(1)}_{ij}\right) \\ &= \bar{g}_0^{ij}\partial_i\bar{F}^{(1)}_{N_0 j} - \frac{1}{2}N_0\mathrm{tr}_{\bar{g}_0}\bar{F}^{(1)} - \bar{g}_0^{ij}\bar{F}^{(1)}_{\ell j}\partial_i N_0^\ell + \frac{1}{2}\bar{F}^{(1)}_{\ell j}N_0\bar{g}_0^{\ell j} \\ &= \bar{F}^{(1)}_{\ell j}\left(-\bar{g}_0^{ij}\partial_i N_0^\ell + \frac{1}{2}N_0\bar{g}_0^{\ell j}\right).\end{aligned} \tag{2.4.18}$$

Since the scalar curvature contains first derivatives of the Christoffel symbols, $R(\gamma)$ contains *a priori* a λ^{-1} contribution from $\partial_\theta\tilde{\Gamma}^{(0)}$, but thanks to the properties of $\bar{F}^{(1)}$ we can see that it vanishes. Indeed we have

$$R^{(-1)} = \bar{g}_0^{ij}\left(\partial_k u_0\partial_\theta(\tilde{\Gamma}^{(0)})^k_{ij} - \partial_i u_0 \partial_\theta(\tilde{\Gamma}^{(0)})^k_{jk}\right)$$

and $\bar{F}^{(1)}_{N_0 i} = 0$ and $\mathrm{tr}_{\bar{g}_0}\bar{F}^{(1)} = 0$ imply that

$$\bar{g}_0^{ij}(\tilde{\Gamma}^{(0)})^k_{ij} = 0 \quad \text{and} \quad (\tilde{\Gamma}^{(0)})^k_{jk} = 0. \tag{2.4.19}$$

Let us now look at the λ^0 terms in $R(\gamma)$. Using again (2.4.19) we obtain:

$$\begin{aligned}R^{(0)} - R(\bar{g}_0) &= \bar{g}_0^{ij}\left(\partial_k u_0\partial_\theta(\Gamma^{(1)})^k_{ij} - \partial_i u_0\partial_\theta(\Gamma^{(1)})^k_{jk}\right) \\ &\quad + \bar{g}_0^{ij}\partial_k(\tilde{\Gamma}^{(0)})^k_{ij} - 2\bar{g}_0^{ij}(\tilde{\Gamma}^{(0)})^k_{i\ell}\Gamma(\bar{g}_0)^\ell_{jk} \\ &\quad - \cos\left(\frac{u_0}{\lambda}\right)(\bar{F}^{(1)})^{ij}\left(\partial_k u_0\partial_\theta(\tilde{\Gamma}^{(0)})^k_{ij} - \partial_i u_0\partial_\theta(\tilde{\Gamma}^{(0)})^k_{jk}\right) - \bar{g}_0^{ij}(\tilde{\Gamma}^{(0)})^k_{i\ell}(\tilde{\Gamma}^{(0)})^\ell_{jk} \\ &= \bar{g}_0^{ij}\left(\partial_k u_0\partial_\theta(\Gamma^{(1)})^k_{ij} - \partial_i u_0\partial_\theta(\Gamma^{(1)})^k_{jk}\right) \\ &\quad + \bar{g}_0^{ij}\partial_k(\tilde{\Gamma}^{(0)})^k_{ij} - 2\bar{g}_0^{ij}(\tilde{\Gamma}^{(0)})^k_{i\ell}\Gamma(\bar{g}_0)^\ell_{jk} \\ &\quad - \cos\left(\frac{u_0}{\lambda}\right)(\bar{F}^{(1)})^{ij}\partial_k u_0\partial_\theta(\tilde{\Gamma}^{(0)})^k_{ij} - \bar{g}_0^{ij}(\tilde{\Gamma}^{(0)})^k_{i\ell}(\tilde{\Gamma}^{(0)})^\ell_{jk}\end{aligned}$$

where we used $(\bar{F}^{(1)})^{ij}\partial_i u_0 = 0$. We set

$$\begin{aligned} I &:= \bar{g}_0^{ij}\left(\partial_k u_0 \partial_\theta (\Gamma^{(1)})^k_{ij} - \partial_i u_0 \partial_\theta (\Gamma^{(1)})^k_{jk}\right), \\ II &:= \bar{g}_0^{ij}\partial_k (\tilde{\Gamma}^{(0)})^k_{ij} - 2\bar{g}_0^{ij}(\tilde{\Gamma}^{(0)})^k_{i\ell}\Gamma(\bar{g}_0)^\ell_{jk}, \\ III &:= -\cos\left(\frac{u_0}{\lambda}\right)(\bar{F}^{(1)})^{ij}\partial_k u_0 \partial_\theta (\tilde{\Gamma}^{(0)})^k_{ij} - \bar{g}_0^{ij}(\tilde{\Gamma}^{(0)})^k_{i\ell}(\tilde{\Gamma}^{(0)})^\ell_{jk}, \end{aligned}$$

and compute further each of these terms. The term I contains all the contributions from $\Gamma^{(1)}$ and (2.4.16) implies

$$\begin{aligned} I &= \sin\left(\frac{u_0}{\lambda}\right)|\nabla u_0|^2_{\bar{g}_0}\left(\omega^{(2)}_{\mathbf{11}} + \omega^{(2)}_{\mathbf{22}}\right) - \frac{1}{2}\cos\left(\frac{2u_0}{\lambda}\right)|\nabla u_0|^2_{\bar{g}_0}\left|\bar{F}^{(1)}\right|^2_{\bar{g}_0} \\ &\quad - \sin\left(\frac{u_0}{\lambda}\right)\bar{g}_0^{ij}\left(\partial_k u_0 \bar{Q}^k_{ij} - \partial_i u_0 \bar{Q}^k_{jk}\right) \\ &= -4\cos\left(\frac{2u_0}{\lambda}\right)|\nabla u_0|^2_{\bar{g}_0}F_0^2 + \sin\left(\frac{u_0}{\lambda}\right)|\nabla u_0|_{\bar{g}_0}\bar{F}^{(1)}_{\ell j}\left(-\bar{g}_0^{ij}\partial_i N_0^\ell + \frac{1}{2}N_0\bar{g}_0^{\ell j}\right) \end{aligned}$$

where we used that the diagonal coefficients of $\omega^{(2)}$ in the frame $(N_0, e_{\mathbf{1}}, e_{\mathbf{2}})$ vanish in addition to (2.4.17)-(2.4.18) and (2.2.12). The term II contains all the linear terms in $\tilde{\Gamma}^{(0)}$. Note that in the notation $\partial_k(\tilde{\Gamma}^{(0)})^k_{ij}$ the derivative does not hit the oscillating part of $\tilde{\Gamma}^{(0)}$ (i.e $\sin\left(\frac{u_0}{\lambda}\right)$ in (2.4.15)). Using (2.4.19) we obtain $\bar{g}_0^{ij}\partial_k(\tilde{\Gamma}^{(0)})^k_{ij} = -(\tilde{\Gamma}^{(0)})^k_{ij}\partial_k\bar{g}_0^{ij}$ and

$$II = -(\tilde{\Gamma}^{(0)})^k_{i\ell}\left(\partial_k\bar{g}_0^{i\ell} + 2\bar{g}_0^{ij}\Gamma(\bar{g}_0)^\ell_{jk}\right) = 0$$

where we used $(\tilde{\Gamma}^{(0)})^k_{i\ell} = (\tilde{\Gamma}^{(0)})^k_{\ell i}$. The term III contains quadratic terms in $\tilde{\Gamma}^{(0)}$. Using (2.4.15) and (2.2.12) we obtain

$$\begin{aligned} III &= -\cos\left(\frac{u_0}{\lambda}\right)(\bar{F}^{(1)})^{ij}\partial_k u_0 \partial_\theta(\tilde{\Gamma}^{(0)})^k_{ij} - \bar{g}_0^{ij}(\tilde{\Gamma}^{(0)})^k_{i\ell}(\tilde{\Gamma}^{(0)})^\ell_{jk} \\ &= -\frac{1}{2}\cos^2\left(\frac{u_0}{\lambda}\right)|\nabla u_0|^2_{\bar{g}_0}\left|\bar{F}^{(1)}\right|^2_{\bar{g}_0} + \frac{1}{4}\sin^2\left(\frac{u_0}{\lambda}\right)|\nabla u_0|^2_{\bar{g}_0}\left|\bar{F}^{(1)}\right|^2_{\bar{g}_0} \\ &= -|\nabla u_0|^2_{\bar{g}_0}F_0^2 - 3\cos\left(\frac{2u_0}{\lambda}\right)|\nabla u_0|^2_{\bar{g}_0}F_0^2. \end{aligned}$$

We conclude the proof of (2.4.10) by adding I and III.

The proof of (2.4.12) comes from the schematic formula

$$R(\gamma) = \gamma^{-2}\partial^2\gamma + \gamma^{-3}(\partial\gamma)^2$$

which gives schematically

$$\begin{aligned} R^{(1)} &= \left((\gamma^{-1})^{(2)}(\gamma^{-1})^{(0)} + \left((\gamma^{-1})^{(1)}\right)^2\right)(\partial^2\gamma)^{(-1)} + (\gamma^{-1})^{(1)}(\gamma^{-1})^{(0)}(\partial^2\gamma)^{(0)} \\ &\quad + \left((\gamma^{-1})^{(0)}\right)^2(\partial^2\gamma)^{(1)} + (\gamma^{-1})^{(1)}\left((\gamma^{-1})^{(0)}(\partial\gamma)^{(0)}\right)^2 + \left((\gamma^{-1})^{(0)}\right)^3(\partial\gamma)^{(1)}(\gamma^{-1})^{(0)} \end{aligned} \tag{2.4.20}$$

where we recall that our high-frequency notations introduced in Section 2.1.2.4 gives for instance

$$(\partial^2\gamma)^{(1)} = \cos\left(\frac{u_0}{\lambda}\right)\partial^2\bar{F}^{(1)} + 2\cos\left(\frac{u_0}{\lambda}\right)\partial\omega^{(2)}$$

and

$$(\gamma^{-1})^{(1)} = -\cos\left(\frac{u_0}{\lambda}\right)\bar{F}^{(1)}.$$

Using (2.4.1) and (2.4.13) we obtain

$$\begin{aligned}(\gamma^{-1})^{(0)} &\overset{\text{osc}}{\sim} 1,\\ (\gamma^{-1})^{(1)} &\overset{\text{osc}}{\sim} \cos(\theta),\\ (\gamma^{-1})^{(2)} &\overset{\text{osc}}{\sim} 1+\sin(\theta)+\cos(2\theta)\end{aligned}$$

and

$$\begin{aligned}(\partial\gamma)^{(0)},\ (\partial^2\gamma)^{(0)} &\overset{\text{osc}}{\sim} 1+\sin(\theta),\\ (\partial\gamma)^{(1)},\ (\partial^2\gamma)^{(-1)},\ (\partial^2\gamma)^{(1)} &\overset{\text{osc}}{\sim} \cos(\theta).\end{aligned}$$

Using these oscillating behaviours we can prove by a direct computation that $R^{(1)}$ defined by (2.4.20) satisfies (2.4.12). This concludes the proof of the lemma. □

Remark 2.4.2. *Note that we denote by γ^{-k} any product of k coefficients of the inverse metric γ^{-1}. This also applies to the background inverse metric $\bar{g}_0^{-1}$.*

2.4.2 Expansion of differential operators

The equations (2.3.3) and (2.3.4) involve differential operators depending on the metric γ. When they are applied to high-frequency quantities, we need to take into account the expansion (2.4.1) affecting the coefficients of the operators as well as the expansion of the quantities themselves. In this section we compute such expansions for all the operators involved, that is the Laplace-Beltrami operator, the divergence operator, the conformal Killing operator and the conformal Laplacian.

In terms of notation, if P_γ is a differential operator acting on tensors of any type and whose coefficients depends on γ, we can formally obtain an expansion of $P_\gamma(T)$ of the form

$$P_\gamma(T) = \sum_k \lambda^k P_\gamma^{[k]}(T).$$

where the previous sum has finite support. The bracket notation $[k]$ thus plays the same role for differential operators as the parenthesis notation (i) introduced in Section 2.1.2.4 for tensors.

Moreover we can mix the two cases, i.e apply differential operators P_γ depending on γ to oscillating tensors $T\left(\frac{u_0}{\lambda}\right)$. The expansion of $P_\gamma(T)$ then depends on the order of $T \longmapsto P_\gamma(T)$ and also $\gamma \longmapsto P_\gamma(T)$. By the order of $\gamma \longmapsto P_\gamma(T)$, we simply mean the top derivative of γ appearing in the coefficients of P_γ. Since both oscillating and non-oscillating terms appear in the expansions for the parameters and the unknowns of the conformal method (see Section 2.3.1), we make a difference between the expansions for $P_\gamma(T)$ when T is non-oscillating and when T is oscillating. Usual capital letters are used in the first case and bold capital letters in the second case, i.e

$$P_\gamma(T) = \sum_k \lambda^k P_\gamma^{[k]}(T) \quad \text{and} \quad P_\gamma\left(T\left(\frac{u_0}{\lambda}\right)\right) = \sum_k \lambda^k \mathbf{P}_\gamma^{[k]}(T)$$

where the support of the two previous finite sums are *a priori* different, depending on T and P_γ. This explains the difference between Lemmas 2.4.6 and 2.4.7 below.

2.4.2.1 The Laplace-Beltrami operator

The only differential operator in the hamiltonian constraint (2.3.3) is the Laplace-Beltrami operator associated to γ. If h is a generic Riemannian metric on Σ_0, we define its Laplace-Beltrami operator by

$$\Delta_h f = h^{ij}\left(\partial_i\partial_j f - \Gamma(h)^k_{ij}\partial_k f\right)$$

for f a scalar function. Note that Δ_e is the usual Laplacian operator on $\mathbb{R}^3$ and is denoted by Δ in this chapter. Since $\gamma = O\left(\lambda^0\right)$ and $\partial\gamma = O\left(\lambda^0\right)$, if f does not admit a high-frequency expansion then

$$\Delta_\gamma f = O\left(\lambda^0\right).$$

If f admit a high-frequency expansion we have the following lemma.

Lemma 2.4.3. *For $f\left(\frac{u_0}{\lambda}\right)$ an oscillating scalar function we have*

$$\Delta_\gamma\left(f\left(\frac{u_0}{\lambda}\right)\right) = \frac{1}{\lambda^2}\mathbf{H}^{[-2]}(f) + \frac{1}{\lambda}\mathbf{H}^{[-1]}(f) + \mathbf{H}^{[\geq 0]}(f)$$

with

$$\mathbf{H}^{[-2]}(f) = |\nabla u_0|^2_{\bar{g}_0}\partial^2_\theta f, \tag{2.4.21}$$
$$\mathbf{H}^{[-1]}(f) = 2\bar{g}_0^{ij}\partial_i u_0\partial_j\partial_\theta f + (\tilde{\Delta}_{\bar{g}_0}u_0)\partial_\theta f - \bar{g}_0^{ij}\Gamma(\bar{g}_0)^k_{ij}\partial_k u_0\partial_\theta f. \tag{2.4.22}$$

with

$$\left|\mathbf{H}^{[\geq 0]}(f)\right| \lesssim \left|\gamma^{-1}\partial^2 f\right| + \left|\gamma^{-2}\partial\gamma\partial f\right|. \tag{2.4.23}$$

Proof. We start with the definition of the Laplace-Beltrami operator :

$$\Delta_\gamma\left(\varphi\left(\frac{u_0}{\lambda}\right)\right) = \gamma^{ij}\partial_i\partial_j\left(\varphi\left(\frac{u_0}{\lambda}\right)\right) - \gamma^{ij}\Gamma(\gamma)^k_{ij}\partial_k\left(\varphi\left(\frac{u_0}{\lambda}\right)\right).$$

Using the expansion of the inverse of γ and $\bar{F}^{(1)}_{N_0 i} = 0$ we have

$$\begin{aligned}\gamma^{ij}\partial_i\partial_j\left(\varphi\left(\frac{u_0}{\lambda}\right)\right) &= \frac{1}{\lambda^2}|\nabla u_0|^2_\gamma\partial^2_\theta\varphi + \frac{1}{\lambda}\left(2\gamma^{ij}\partial_i u_0\partial_j\partial_\theta\varphi + (\tilde{\Delta}_\gamma u_0)\partial_\theta\varphi\right) + O\left(\lambda^0\right)\\ &= \frac{1}{\lambda^2}|\nabla u_0|^2_{\bar{g}_0}\partial^2_\theta\varphi + \frac{1}{\lambda}\left(2\bar{g}_0^{ij}\partial_i u_0\partial_j\partial_\theta\varphi + (\tilde{\Delta}_{\bar{g}_0}u_0)\partial_\theta\varphi\right) + O\left(\lambda^0\right)\end{aligned}$$

where $\tilde{\Delta}_h = h^{ij}\partial_i\partial_j$. Moreover from the decomposition of the Christoffel symbols (2.4.14) and (2.4.19) we obtain

$$\gamma^{ij}\Gamma(\gamma)^k_{ij}\partial_k\left(\varphi\left(\frac{u_0}{\lambda}\right)\right) = \frac{1}{\lambda}\bar{g}_0^{ij}\Gamma(\bar{g}_0)^k_{ij}\partial_k u_0\partial_\theta\varphi + O(\lambda).$$

The estimate (2.4.23) simply comes from the definition of Δ_γ. ☐

2.4.2.2 The divergence operator

The divergence operator appears in the momentum constraint (2.3.4) but also in the definition of the parameter σ, which in particular needs to be a divergence free tensor for the metric γ. Recall that if h is a Riemannian metric on Σ_0 and if $\mathbf{D}^{(h)}$ denotes the covariant derivative associated, then $\mathrm{div}_h A = h^{k\ell}\mathbf{D}^{(h)}_k A_\ell$ and $(\mathrm{div}_h B)_i = h^{k\ell}\mathbf{D}^{(h)}_k B_{\ell i}$ for A a 1-tensor and B a 2-tensor. The divergence operator only depends on first derivatives of γ which are $O\left(\lambda^0\right)$ so we only need an expansion when the tensor on which it acts is itself oscillating.

Lemma 2.4.4. *For $A_{ij}\left(\frac{u_0}{\lambda}\right)$ an oscillating symmetric 2-tensor we have*

$$\mathrm{div}_\gamma A\left(\frac{u_0}{\lambda}\right)_\ell = \frac{1}{\lambda}\mathbf{d}^{[-1]}_\ell(A) + \mathbf{d}^{[0]}_\ell(A) + \lambda\mathbf{d}^{[\geq 1]}_\ell(A)$$

with

$$\mathbf{d}^{[-1]}_\ell(A) = -|\nabla u_0|_{\bar{g}_0}\partial_\theta A_{N_0\ell}, \tag{2.4.24}$$

$$\mathbf{d}^{[0]}_\ell(A) = \mathrm{div}_{\bar{g}_0}A_\ell - (\bar{g}_0)^{ij}(\tilde{\Gamma}^{(0)})^a_{i\ell}A_{aj}, \tag{2.4.25}$$

and

$$\left|\mathbf{d}^{[\geq 1]}_\ell(A)\right| \lesssim \left|\gamma^{-1}\partial A\right| + \left|\gamma^{-2}\partial\gamma A\right|. \tag{2.4.26}$$

Moreover

$$\mathbf{d}^{[0]}(A) \overset{\mathrm{osc}}{\sim} (1+\sin(\theta))A, \tag{2.4.27}$$

$$\mathbf{d}^{[1]}(A) \overset{\mathrm{osc}}{\sim} \sin(\theta)\partial_\theta A + (\cos(\theta)+\sin(2\theta))A. \tag{2.4.28}$$

Proof. We start by using the expansion of the Christoffel's symbols from (2.4.14):

$$\mathbf{D}^{(\gamma)}_i A\left(\frac{u_0}{\lambda}\right)_{j\ell} = \frac{1}{\lambda}\partial_i u_0\partial_\theta A_{j\ell} + \bar{\mathbf{D}}_i A_{j\ell} - (\tilde{\Gamma}^{(0)})^a_{i(j}A_{a\ell)} - \lambda(\Gamma^{(1)})^a_{i(j}A_{a\ell)} + O\left(\lambda^2\right)$$

We now use the expansion of the inverse of γ:

$$\begin{aligned}\mathrm{div}_\gamma A\left(\frac{u_0}{\lambda}\right)_\ell &= \gamma^{ij}\mathbf{D}^{(\gamma)}_i A_{j\ell}\\ &= -\frac{1}{\lambda}|\nabla u_0|_{\bar{g}_0}\partial_\theta A_{N_0\ell} + \mathrm{div}_{\bar{g}_0}A_\ell - \bar{g}_0^{ij}(\tilde{\Gamma}^{(0)})^a_{i(j}A_{a\ell)} - \cos\left(\frac{u_0}{\lambda}\right)(\bar{F}^{(1)})^{ij}\partial_i u_0\partial_\theta A_{j\ell} \qquad (2.4.29)\\ &\quad + \lambda\Big[\cos^2\left(\frac{u_0}{\lambda}\right)\bar{g}_0^{ik}(\bar{F}^{(1)})^{ja}\bar{F}^{(1)}_{ka}\partial_i u_0\partial_\theta A_{j\ell} - \sin\left(\frac{u_0}{\lambda}\right)(\omega^{(2)})^{ij}\partial_i u_0\partial_\theta A_{j\ell}\\ &\qquad - \cos\left(\frac{u_0}{\lambda}\right)(\bar{F}^{(1)})^{ij}\left(\bar{\mathbf{D}}_i A_{j\ell} - (\tilde{\Gamma}^{(0)})^a_{i(j}A_{a\ell)}\right) - \bar{g}_0^{ij}(\Gamma^{(1)})^a_{i(j}A_{a\ell)}\Big]\\ &\quad + O\left(\lambda^2\right)\end{aligned}$$

This concludes the proof, using $\bar{F}^{(1)}_{N_0 i} = 0$ and $\bar{g}_0^{ij}(\tilde{\Gamma}^{(0)})^a_{ij} = 0$. The estimate (2.4.26) just comes from the definition of the divergence. The oscillating behaviours (2.4.27) and (2.4.28) can be directly read on (2.4.29). □

2.4.2.3 The conformal Killing operator

In order to compute $L_\gamma W$ in the ansatz for K (2.3.2), we need to expand the conformal Killing operator defined by

$$(L_h A)_{ij} = \mathbf{D}^{(h)}_{(i} A_{j)} - \frac{2}{3}(\mathrm{div}_h A) h_{ij} \tag{2.4.30}$$

where A is a 1-tensor. Note that $L_h A$ is a symmetric 2-tensor traceless with respect to h. As for the divergence operator, L_γ only depends on first derivatives of γ so only an expansion of $L_\gamma W$ with W oscillating is necessary.

Lemma 2.4.5. *Let $W\left(\frac{u_0}{\lambda}\right)$ be an oscillating 1-form. We have*

$$L_\gamma\left(W\left(\frac{u_0}{\lambda}\right)\right)_{ij} = \frac{1}{\lambda}\mathbf{K}^{[-1]}_{ij}(W) + \mathbf{K}^{[0]}_{ij}(W) + \lambda \mathbf{K}^{[\geq 1]}_{ij}(W)$$

where

$$\mathbf{K}^{[-1]}_{ij}(W) = \partial_{(i} u_0 \partial_\theta W_{j)} + \frac{2}{3}|\nabla u_0|_{\bar{g}_0} (\bar{g}_0)_{ij} \partial_\theta W_{N_0}, \tag{2.4.31}$$

$$\mathbf{K}^{[0]}_{ij}(W) = \bar{\mathbf{D}}_{(i} W_{j)} - 2W_k (\tilde{\Gamma}^{(0)})^k_{ij} + \frac{2}{3}\cos\left(\frac{u_0}{\lambda}\right)|\nabla u_0|_{\bar{g}_0} \bar{F}^{(1)}_{ij} \partial_\theta W_{N_0} - \frac{2}{3}\mathrm{div}_{\bar{g}_0} W (\bar{g}_0)_{ij}, \tag{2.4.32}$$

and

$$\left|\mathbf{K}^{[\geq 1]}_{ij}(W)\right| \lesssim |\partial W| + \left|\gamma^{-1}\partial\gamma W\right|. \tag{2.4.33}$$

The following hold:

$$\mathrm{tr}_{\bar{g}_0}\mathbf{K}^{[-1]}(W) = 0, \tag{2.4.34}$$

$$\mathrm{tr}_{\bar{g}_0}\mathbf{K}^{[0]}(W) = \cos\left(\frac{u_0}{\lambda}\right)\mathrm{tr}_{\bar{F}^{(1)}}\mathbf{K}^{[-1]}(W). \tag{2.4.35}$$

The following oscillating behaviour holds

$$\mathbf{K}^{[0]}(W) \overset{\mathrm{osc}}{\sim} (1+\sin(\theta))W + \cos(\theta)\partial_\theta W. \tag{2.4.36}$$

Proof. We start with the covariant derivative of the oscillating 1-form $W^{(\cdot)}\left(\frac{u_0}{\lambda}\right)$:

$$\mathbf{D}^{(\gamma)}_i\left(W\left(\frac{u_0}{\lambda}\right)\right)_j = \frac{1}{\lambda}\partial_i u_0 \partial_\theta W_j + \bar{\mathbf{D}}_i W_j\left(\frac{u_0}{\lambda}\right) - W_k\left((\tilde{\Gamma}^{(0)})^k_{ij} + \lambda(\Gamma^{(1)})^k_{ij} + \lambda^2(\Gamma^{(2)})^k_{ij} + \cdots\right)$$

which also gives us the divergence

$$\begin{aligned}\mathrm{div}_\gamma\left(W\left(\frac{u_0}{\lambda}\right)\right) &= \gamma^{ij}\mathbf{D}^{(\gamma)}_i\left(W\left(\frac{u_0}{\lambda}\right)\right)_j \\ &= \frac{1}{\lambda}(\bar{g}_0)^{k\ell}\partial_k u_0 \partial_\theta W_\ell - \cos\left(\frac{u_0}{\lambda}\right)(\bar{F}^{(1)})^{k\ell}\partial_k u_0 \partial_\theta W_\ell \\ &\quad + \mathrm{div}_{\bar{g}_0} W - (\bar{g}_0)^{k\ell}(\tilde{\Gamma}^{(0)})^a_{k\ell} W_a + O(\lambda)\end{aligned}$$

We conclude the proof of (2.4.31)-(2.4.32) with

$$L_\gamma\left(W\left(\frac{u_0}{\lambda}\right)\right)_{ij} = \mathbf{D}^{(\gamma)}_{(i}\left(W\left(\frac{u_0}{\lambda}\right)\right)_{j)} - \frac{2}{3}\mathrm{div}_\gamma\left(W\left(\frac{u_0}{\lambda}\right)\right)\gamma_{ij}$$

and $\bar{F}^{(1)}_{N_0 i} = 0$ and $(\bar{g}_0)^{ij}(\tilde{\Gamma}^{(0)})^a_{ij} = 0$. The trace identities (2.4.34)-(2.4.35) and the oscillating behaviour (2.4.36) follows directly from (2.4.31)-(2.4.32). □

Remark 2.4.3. *In the previous lemma we used the notation $\mathrm{tr}_{\bar{F}^{(1)}}\mathbf{K}^{[-1]}(W)$ to denote*

$$(\bar{F}^{(1)})^{ij}\mathbf{K}^{[-1]}_{ij}(W) = \bar{g}_0^{ik}\bar{g}_0^{j\ell}\bar{F}^{(1)}_{k\ell}\mathbf{K}^{[-1]}_{ij}(W)$$

even though $\bar{F}^{(1)}$ is not a Lorentzian metric.

2.4.2.4 The conformal Laplacian

The conformal Laplacian associated to a Riemannian metric h is the operator $\mathrm{div}_h L_h$ acting on 1-tensor. It is the only operator considered in this chapter depending on $\partial^2\gamma$, i.e second derivatives of the metric γ. Indeed L_γ contains the Christoffel symbols of γ through the covariant derivative, and they are differenciated once by the divergence. Since $\partial^2\gamma = O\left(\lambda^{-1}\right)$, this a major difference with the Laplace-Beltrami operator or the divergence operator from the point of view of expanding quantities in powers of λ. Indeed, this implies that even if the 1-form β is not oscillating, the quantity $\mathrm{div}_\gamma L_\gamma \beta$ still loses one power of λ. If β is oscillating, then $\mathrm{div}_\gamma L_\gamma \beta$ loses two powers of λ since the conformal Laplacian is a second order operator. This explains the two following lemmas.

Lemma 2.4.6. *Let β be a 1-form. We have*

$$\mathrm{div}_\gamma L_\gamma \beta = \frac{1}{\lambda} M^{[-1]}(\beta) + M^{[\geq 0]}(\beta)$$

with

$$M_\ell^{[-1]}(\beta) = |\nabla u_0|^2_{\bar{g}_0} \cos\left(\frac{u_0}{\lambda}\right) \bar{g}_0^{ij} \bar{F}^{(1)}_{i\ell} \beta_j \tag{2.4.37}$$

and

$$\begin{aligned} \left| M^{[\geq 0]}(\beta) - \mathrm{div}_e L_e \beta \right| &\lesssim \left|\gamma^{-1} - e^{-1}\right| |\partial^2 \beta| + \left|\gamma^{-2}\partial\gamma\right| \left|1 + \gamma^{-1}\gamma\right| |\partial\beta| \\ &\quad + \left|\gamma^{-3}(\partial\gamma)^2 + \gamma^{-2}(\partial^2\gamma)^{(\geq 0)}\right| \left|1 + \gamma^{-1}\gamma\right| |\beta|. \end{aligned} \tag{2.4.38}$$

Proof. Let $\bar{g}$ be any Riemannian metric on $\mathbb{R}^3$, we have

$$\begin{aligned} \mathrm{div}_{\bar{g}} L_{\bar{g}} \beta_\ell = {} & \bar{g}^{ij}\left(\partial_i \partial_{(j}\beta_{\ell)} - \frac{2}{3}\partial_\ell \partial_i \beta_j\right) + \beta_k \left(-2\bar{g}^{ij}\partial_i \Gamma(\bar{g})^k_{j\ell} + \frac{2}{3}\partial_\ell\left(\bar{g}^{ij}\Gamma(\bar{g})^k_{ij}\right)\right) \\ & - \bar{g}^{ij}\left(2\Gamma(\bar{g})^k_{j\ell}\partial_i\beta_k + \frac{2}{3}\mathrm{div}_{\bar{g}}\beta \partial_i \bar{g}_{j\ell}\right) + \frac{2}{3}\left(-\partial_i\beta_j\partial_\ell \bar{g}^{ij} + \bar{g}^{ij}\Gamma(\bar{g})^k_{ij}\partial_\ell\beta_k\right) \\ & - \bar{g}^{ij}\Gamma(\bar{g})^k_{i(j} L_{\bar{g}}\beta_{k\ell)}. \end{aligned} \tag{2.4.39}$$

In the case of the high-frequency metric γ, the only terms loosing $\frac{1}{\lambda}$, i.e contributing to $M^{[-1]}(\beta)$, are the $\gamma^{-2}\partial^2\gamma\beta$. More precisely, it concerns terms involving $\partial\tilde{\Gamma}^{(0)}$, and since $\bar{g}_0^{ij}(\tilde{\Gamma}^{(0)})^k_{ij} = 0$, the only contribution is

$$M_\ell^{[-1]}(\beta) = -2\beta_k \bar{g}^{ij}\partial_i u_0 \partial_\theta (\tilde{\Gamma}^{(0)})^k_{j\ell} = |\nabla u_0|^2_{\bar{g}_0}\cos\left(\frac{u_0}{\lambda}\right)\bar{g}_0^{ij}\bar{F}^{(1)}_{i\ell}\beta_j$$

where we used (2.4.15) and $\bar{F}^{(1)}_{N_0 i} = 0$. From (2.4.39) we also get

$$\mathrm{div}_e L_e \beta_\ell = e^{ij}\left(\partial_i\partial_{(j}\beta_{\ell)} - \frac{2}{3}\partial_\ell\partial_i\beta_j\right)$$

which concludes the proof of (2.4.38). □

The estimation (2.4.38) will allow us to invert the operator $M^{[\geq 0]}$, since we know how to invert $\mathrm{div}_e L_e$ (see Proposition 2.1.2) and since we gain a smallness constant ε in front of $\partial^2\beta$ thanks to (2.2.4) and (2.4.13). We now state the final lemma of this section, which will allow us to construct high-frequency solutions of the momentum constraint. The proof is left to the reader since it follows from Lemmas 2.4.4 and 2.4.5.

Lemma 2.4.7. *Let $W\left(\frac{u_0}{\lambda}\right)$ be an oscillating 1-form. We have*

$$\mathrm{div}_\gamma L_\gamma\left(W\left(\frac{u_0}{\lambda}\right)\right)_\ell = \frac{1}{\lambda^2}\mathbf{M}_\ell^{[-2]}(W) + \frac{1}{\lambda}\mathbf{M}_\ell^{[-1]}(W) + \mathbf{M}_\ell^{[\geq 0]}(W)$$

with

$$\mathbf{M}_\ell^{[-2]}(W) = \mathbf{d}_\ell^{[-1]}\left(\mathbf{K}^{[-1]}(W)\right), \tag{2.4.40}$$

$$\mathbf{M}_\ell^{[-1]}(W) = \mathbf{d}_\ell^{[-1]}\left(\mathbf{K}^{[0]}(W)\right) + \mathbf{d}_\ell^{[0]}\left(\mathbf{K}^{[-1]}(W)\right), \tag{2.4.41}$$

The following oscillating behaviour holds

$$\mathbf{M}^{[-1]}(W) \sim \cos(\theta)W + \cos(\theta)\partial_\theta^2 W + (1+\sin(\theta))\partial_\theta W. \tag{2.4.42}$$

From (2.4.24), (2.4.31) and (2.4.40) we obtain

$$\mathbf{M}_\ell^{[-2]}(W) = |\nabla u_0|^2_{\bar{g}_0}\left(\partial_\theta^2 W_\ell + \frac{1}{3}(N_0)_\ell \partial_\theta^2 W_{N_0}\right). \tag{2.4.43}$$

In order to solve the momentum constraint, we need to invert this operator. This is done in the following simple lemma.

Lemma 2.4.8. *If α and β are 1-forms such that*

$$\alpha_\ell + \frac{1}{3}(N_0)_\ell \alpha_{N_0} = \beta_\ell \tag{2.4.44}$$

then

$$\alpha_\ell = \beta_\ell - \frac{1}{4}(N_0)_\ell \beta_{N_0}.$$

Proof. We contract (2.4.44) with the vector field N_0 to obtain

$$\frac{4}{3}\alpha_{N_0} = \beta_{N_0}.$$

Inserting this into (2.4.44) concludes the proof. □

2.5 Approximate solution to the constraint equations

In this section, we construct an approximate solution to the constraint equations (2.3.3) and (2.3.4) by solving the λ^0 and λ^1 Hamiltonian level and the λ^0 momentum level. In the process we will define the parameter τ and in Sections 2.5.3 and 2.5.4 we construct the parameter σ.

2.5.1 The approximate Hamiltonian constraint

We solve the first two levels of the Hamiltonian constraint by choosing $\varphi^{(2)}$ and $\varphi^{(3)}$. At those levels, (2.3.3) decouples from (2.3.4) since we replace $(\sigma + L_\gamma W)^{(\leq 1)}$ by $\left(K - \frac{1}{3}\tau\gamma\right)^{(\leq 1)}$, where $K^{(\leq 1)}$ and $\tau^{(\leq 1)}$ will be defined along the way.

2.5.1.1 The λ^0 Hamiltonian level

We want $\varphi^{(2)}$ to solve the λ^0 Hamiltonian level. Using the expansion of the Laplace-Beltrami operator Δ_γ (see Lemma 2.4.3), this is equivalent to the following equation:

$$8|\nabla u_0|^2_{\bar{g}_0}\partial^2_\theta\varphi^{(2)} = R^{(0)} + \frac{2}{3}\left(\tau^{(0)}\right)^2 - \left|K^{(0)} - \frac{1}{3}\tau^{(0)}\bar{g}_0\right|^2_{\bar{g}_0}. \tag{2.5.1}$$

where $K^{(0)}$ is defined by

$$K^{(0)} = K_0 + \frac{1}{2}\sin\left(\frac{u_0}{\lambda}\right)|\nabla u_0|_{\bar{g}_0}\bar{F}^{(1)} \tag{2.5.2}$$

so that the compatibility with the spacetime ansatz is satisfied. As the parameter τ is the trace with respect to γ of K, we have

$$\begin{aligned}\tau^{(0)} &= \mathrm{tr}_{\bar{g}_0}K^{(0)}\\ &= \mathrm{tr}_{\bar{g}_0}K_0 + \frac{1}{2}\sin\left(\frac{u_0}{\lambda}\right)|\nabla u_0|_{\bar{g}_0}\mathrm{tr}_{\bar{g}_0}\bar{F}^{(1)}\\ &= 0\end{aligned} \tag{2.5.3}$$

where we used (2.2.3) and Lemma 2.4.1. Therefore, (2.5.1) rewrites

$$8|\nabla u_0|^2_{\bar{g}_0}\partial^2_\theta\varphi^{(2)} = R^{(0)} - \left|K^{(0)}\right|^2_{\bar{g}_0}. \tag{2.5.4}$$

Since the LHS of (2.5.4) is a derivative with respect to θ, we need the RHS to be purely oscillating. Since this shows how the creation of non-oscillating terms is absorbed by the background constraint equations we state this in a separate lemma.

Lemma 2.5.1. *We have*

$$\begin{aligned}R^{(0)} - \left|K^{(0)}\right|^2_{\bar{g}_0} &= \sin\left(\frac{u_0}{\lambda}\right)|\nabla u_0|_{\bar{g}_0}\bar{F}^{(1)}_{\ell j}\left(-\bar{g}_0^{ij}\partial_i N_0^\ell + \frac{1}{2}\left(N_0 - \partial_t\right)\bar{g}_0^{\ell j}\right)\\ &\quad - 6\cos\left(\frac{2u_0}{\lambda}\right)|\nabla u_0|^2_{\bar{g}_0}F_0^2.\end{aligned}$$

Proof. Let us first expand $\left|K^{(0)}\right|^2_{\bar{g}_0}$ using (2.5.2) and (2.2.12):

$$\left|K^{(0)}\right|^2_{\bar{g}_0} = |K_0|^2_{\bar{g}_0} + |\nabla u_0|^2_{\bar{g}_0}F_0^2 - \cos\left(\frac{2u_0}{\lambda}\right)|\nabla u_0|^2_{\bar{g}_0}F_0^2 + \sin\left(\frac{u_0}{\lambda}\right)|\nabla u_0|_{\bar{g}_0}\left|\bar{F}^{(1)}\cdot K_0\right|_{\bar{g}_0}.$$

We use now (2.4.10) to compute the full RHS of (2.5.4):

$$\begin{aligned}R^{(0)} - \left|K^{(0)}\right|^2_{\bar{g}_0} &= R(\bar{g}_0) - |K_0|^2_{\bar{g}_0} - 2|\nabla u_0|^2_{\bar{g}_0}F_0^2\\ &\quad + \sin\left(\frac{u_0}{\lambda}\right)|\nabla u_0|_{\bar{g}_0}\left(\bar{F}^{(1)}_{\ell j}\left(-\bar{g}_0^{ij}\partial_i N_0^\ell + \frac{1}{2}N_0\bar{g}_0^{\ell j}\right) - \left|\bar{F}^{(1)}\cdot K_0\right|_{\bar{g}_0}\right)\\ &\quad - 6\cos\left(\frac{2u_0}{\lambda}\right)|\nabla u_0|^2_{\bar{g}_0}F_0^2\end{aligned}$$

where we crucially used the background Hamiltonian constraint (2.2.1) to cancel the non-oscillating term. Using now

$$\left|\bar{F}^{(1)}\cdot K_0\right|_{\bar{g}_0} = \frac{1}{2}\bar{F}^{(1)}_{\ell j}\partial_t\bar{g}_0^{\ell j} \tag{2.5.5}$$

we conclude the proof of the lemma. □

This lemma allows us to solve (2.5.4) by simply setting

$$\varphi^{(2)} = \frac{1}{8|\nabla u_0|_{\bar{g}_0}} \sin\left(\frac{u_0}{\lambda}\right) \bar{F}^{(1)}_{\ell j} \left(\bar{g}_0^{ij} \partial_i N_0^\ell + \frac{1}{2} (\partial_t - N_0) \bar{g}_0^{\ell j} \right) + \frac{3}{16} \cos\left(\frac{2u_0}{\lambda}\right) F_0^2. \tag{2.5.6}$$

Now that $\varphi^{(2)}$ is defined, we can set

$$\bar{F}^{(2,1)} = \frac{\bar{F}^{(1)}_{\ell j}}{2|\nabla u_0|_{\bar{g}_0}} \left(\bar{g}_0^{ij} \partial_i N_0^\ell + \frac{1}{2} (\partial_t - N_0) \bar{g}_0^{\ell j} \right) \bar{g}_0 + \omega^{(2)}, \tag{2.5.7}$$

$$\bar{F}^{(2,2)} = \frac{3}{4} F_0^2 \bar{g}_0. \tag{2.5.8}$$

These definitions are consistent with (2.3.1) and the following lemma summarizes the properties of $\bar{F}^{(2,1)}$ and $\bar{F}^{(2,2)}$:

Lemma 2.5.2. *The symmetric 2-tensors $\bar{F}^{(2,1)}$ and $\bar{F}^{(2,2)}$ are supported in B_R and the following holds*

$$\left\| \bar{F}^{(2,1)} \right\|_{H^{N-1}} + \left\| \bar{F}^{(2,2)} \right\|_{H^{N-1}} \lesssim \varepsilon \tag{2.5.9}$$

with a constant depending only on δ and R.

Proof. The support property follows from the support properties of F_0, $\bar{F}^{(1)}$ and $\omega^{(2)}$. The estimate (2.5.9) follows from the definitions (2.5.7) and (2.5.8) and the estimates (2.2.4), (2.2.6) and (2.4.9) (recall also (2.4.7) for the definition of $\omega^{(2)}$). □

2.5.1.2 The λ^1 Hamiltonian level

We now turn to the λ^1 Hamiltonian level, which is solved thanks to $\varphi^{(3)}$. More precisely, since $\tau = O(\lambda)$ we need $\varphi^{(3)}$ to satisfy

$$\begin{aligned} 8|\nabla u_0|^2_{\bar{g}_0} \partial_\theta^2 \varphi^{(3)} &= -8\mathbf{H}^{[-1]}\left(\varphi^{(2)}\right) + R^{(1)} \\ &\quad - \left| K^{(0)} \cdot \left(K^{(1)} - \frac{1}{3} \tau^{(1)} \bar{g}_0 \right) \right|_{\bar{g}_0} + \cos\left(\frac{u_0}{\lambda}\right) \left| K^{(0)} \right|^2_{\bar{F}^{(1)}} \end{aligned} \tag{2.5.10}$$

where $R^{(1)}$ is defined in Lemma 2.4.2 and $\mathbf{H}^{[-1]}\left(\varphi^{(2)}\right)$ in Lemma 2.4.3. In the RHS of (2.5.10), it remains to define $K^{(1)}$ and $\tau^{(1)}$. In order to satisfy the compatibility with the spacetime ansatz (recall the discussion from Section 2.3.2.2), we set

$$\begin{aligned} K^{(1)}_{ij} = &-\frac{1}{2} \cos\left(\frac{u_0}{\lambda}\right) \left(-N_0 \bar{F}^{(1)}_{ij} + (\partial_t + N_0)^\rho \Gamma(g_0)^k_{\rho(i} \bar{F}^{(1)}_{j)k} + \frac{1}{2|\nabla u_0|_{\bar{g}_0}} (\Box_{g_0} u_0) \bar{F}^{(1)}_{ij} \right) \\ &- \frac{1}{2} |\nabla u_0|_{\bar{g}_0} \left(\cos\left(\frac{u_0}{\lambda}\right) \bar{F}^{(2,1)}_{ij} - 2 \sin\left(\frac{2u_0}{\lambda}\right) \bar{F}^{(2,2)}_{ij} \right). \end{aligned} \tag{2.5.11}$$

Since τ is the trace with respect to γ of K we set

$$\tau^{(1)} = -\cos\left(\frac{u_0}{\lambda}\right) \mathrm{tr}_{\bar{F}^{(1)}} K^{(0)} + \mathrm{tr}_{\bar{g}_0} K^{(1)} \tag{2.5.12}$$

where we used the expansion of the inverse of γ given by (2.4.13) and where we used the notation $\mathrm{tr}_S T = S^{ij} T_{ij}$ for S and T two symmetric 2-tensors with the inverse of S being defined with respect to $\bar{g}_0$. This actually fully defines the parameter τ, i.e

$$\tau = \lambda \tau^{(1)}. \tag{2.5.13}$$

As above, we need to show that the RHS of (2.5.10) is purely oscillating in order to find $\varphi^{(3)}$ solution of this equation. From (2.5.2), (2.5.11) and (2.5.12) we obtain

$$K^{(0)} \overset{\text{osc}}{\sim} 1+\sin(\theta) \quad \text{and} \quad K^{(1)},\, \tau^{(1)} \overset{\text{osc}}{\sim} \cos(\theta)+\sin(2\theta)$$

which together with (2.4.12), (2.4.22) and (2.5.6) gives

$$\text{RHS of (2.5.10)} \overset{\text{osc}}{\sim} \cos(\theta)+\sin(2\theta)+\cos(3\theta).$$

This shows that we can solve (2.5.10) by setting

$$\varphi^{(3)} = \cos\left(\frac{u_0}{\lambda}\right)\varphi^{(3,1)} + \sin\left(\frac{2u_0}{\lambda}\right)\varphi^{(3,2)} + \cos\left(\frac{3u_0}{\lambda}\right)\varphi^{(3,3)} \tag{2.5.14}$$

for some $\varphi^{(3,i)}$ supported in B_R and satisfying

$$\sum_{i=1,2,3}\left\|\varphi^{(3,i)}\right\|_{H^{N-3}} \lesssim \varepsilon. \tag{2.5.15}$$

The estimate (2.5.15) follows from (2.4.11) and (2.4.7). Since the ansatz constructed in this chapter is an order 2 ansatz, the term $\varphi^{(3)}$ will be ultimately hidden in the remainder $\bar{\mathfrak{h}}_\lambda$ (see (2.2.8) in Theorem 2.2.1) and thus we don't need a precise expression of $\varphi^{(3,i)}$.

2.5.2 The approximate momentum constraint

In this section we solve the first level of the momentum constraint. Since $\tau = \lambda\tau^{(1)}$ we have $\mathrm{d}\tau = \frac{1}{\lambda}\mathrm{d}u_0\partial_\theta\tau^{(1)} + \lambda\mathrm{d}\tau^{(1)}$ where the derivatives in $\mathrm{d}\tau^{(1)}$ don't hit the oscillating parts of $\tau^{(1)}$. Therefore the first momentum level corresponds to the λ^0 level of (2.3.4), solved by $W^{(2)}$.

Remark 2.5.1. *This is where the assumption* $\mathrm{tr}_{\bar{g}_0}K_0 = 0$ *simplifies our construction. If* $\mathrm{tr}_{\bar{g}_0}K_0 \neq 0$ *then* τ *needs to include a non-oscillating* λ^0 *term in* $\tau^{(0)}$*. This would require a non-oscillating term in* W *at the* λ^0 *order to absorb it*

$$W = w^{(0)} + O(\lambda)$$

where we use a lowercase letter to emphasize that $w^{(0)}$ *is non-oscillating. However, as explained in Section 2.4.2.4, this term would produce a* λ^{-1} *term in* $\mathrm{div}_\gamma L_\gamma W$ *(precisely* $M^{[-1]}(w^{(0)})$*, see Lemma 2.4.6) and thus would require an oscillating term* $W^{(1)}$ *in* W *to absorb it. We make the assumption* $\mathrm{tr}_{\bar{g}_0}K_0 = 0$ *precisely to avoid these technicalities.*

More precisely, $W^{(2)}$ needs to solve

$$\mathbf{M}_\ell^{[-2]}(W^{(2)}) = \frac{2}{3}\partial_\ell u_0\partial_\theta\tau^{(1)}. \tag{2.5.16}$$

where we used the expansion obtained in Lemma 2.4.7. Thanks to Lemma 2.4.8 this equation rewrites as

$$\partial_\theta^2 W_\ell^{(2)} = -\frac{1}{2|\nabla u_0|_{\bar{g}_0}}(N_0)_\ell\partial_\theta\tau^{(1)} \tag{2.5.17}$$

Since the RHS of this equation is a ∂_θ derivative it is purely oscillating and we can integrate twice to obtain $W^{(2)}$. Using (2.5.12) this also gives

$$W^{(2)} \overset{\text{osc}}{\sim} \sin(\theta)+\cos(2\theta). \tag{2.5.18}$$

Note that as opposed to the Hamiltonian constraint, we don't solve the λ^1 momentum level here. Indeed, as Lemma 2.4.6 shows, the operator $\mathrm{div}_\gamma L_\gamma$ loses one λ power even when applied to a non-oscillating field like $\widetilde{W}$ (the remainder in (2.3.10)). Therefore the λ^1 momentum level also involves $\widetilde{W}$ and the equation for $W^{(3)}$ is coupled with the remainder in the ansatz (2.3.10), as opposed to $\varphi^{(3)}$ which is not coupled to $\widetilde{\varphi}$ (the remainder in (2.3.9)), since $\Delta_\gamma\widetilde{\varphi} = O\left(\lambda^0\right)$.

2.5.3 An almost TT-tensor

In this section we define the first terms in the expansion of the parameter σ of the conformal method, i.e $\sigma^{(0)}$ and $\sigma^{(1)}$. As explained in Section 2.3.2.2, the definition of the first orders of σ follows simply from our constraint

$$(\sigma + L_\gamma W)^{(\leq 1)} = \left(K - \frac{1}{3}\tau\gamma\right)^{(\leq 1)}. \tag{2.5.19}$$

Since W is given by $W = \lambda^2\left(W^{(2)} + \widetilde{W}\right) + O\left(\lambda^3\right)$ (with $W^{(2)}$ defined by (2.5.17) and $\widetilde{W}$ non-oscillating), we use Lemma 2.4.5 to compute $(L_\gamma W)^{(\leq 1)}$ and (2.5.19) forces us to define

$$\sigma^{(0)} = K^{(0)} \tag{2.5.20}$$

$$\sigma^{(1)} = K^{(1)} - \frac{1}{3}\tau^{(1)}\bar{g}_0 - \mathbf{K}^{[-1]}(W^{(2)}) \tag{2.5.21}$$

where $K^{(0)}$, $K^{(1)}$ and $\tau^{(1)}$ are given by (2.5.2), (2.5.11) and (2.5.12) respectively and $\mathbf{K}^{[-1]}(W^{(2)})$ is defined in Lemma 2.4.5.

In this section, we prove that $\sigma^{(0)} + \lambda\sigma^{(1)}$ is *almost* a TT-tensor, that is

$$\mathrm{tr}_\gamma\left(\sigma^{(0)} + \lambda\sigma^{(1)}\right) = O\left(\lambda^2\right) \quad \text{and} \quad \mathrm{div}_\gamma\left(\sigma^{(0)} + \lambda\sigma^{(1)}\right) = O\left(\lambda\right). \tag{2.5.22}$$

Using the expansion of γ^{ij} given by (2.4.13) we have

$$\mathrm{tr}_\gamma\left(\sigma^{(0)} + \lambda\sigma^{(1)}\right) = \mathrm{tr}_{\bar{g}_0}\sigma^{(0)} + \lambda\left(-\cos\left(\frac{u_0}{\lambda}\right)\mathrm{tr}_{\bar{F}^{(1)}}\sigma^{(0)} + \mathrm{tr}_{\bar{g}_0}\sigma^{(1)}\right) + O\left(\lambda^2\right).$$

Using now the expansion of the divergence operator div_γ obtained in Lemma 2.4.4 we have

$$\mathrm{div}_\gamma\left(\sigma^{(0)} + \lambda\sigma^{(1)}\right) = \frac{1}{\lambda}\mathbf{d}^{[-1]}(\sigma^{(0)}) + \mathbf{d}^{[0]}(\sigma^{(0)}) + \mathbf{d}^{[-1]}(\sigma^{(1)}) + O\left(\lambda\right).$$

The following lemma, which shows that $\sigma^{(0)} + \lambda\sigma^{(1)}$ is almost a TT-tensor, is crucial since it validates a posteriori our whole approximate construction.

Lemma 2.5.3. *We have*

$$\mathrm{tr}_{\bar{g}_0}\sigma^{(0)} = 0, \tag{2.5.23}$$

$$-\cos\left(\frac{u_0}{\lambda}\right)\mathrm{tr}_{\bar{F}^{(1)}}\sigma^{(0)} + \mathrm{tr}_{\bar{g}_0}\sigma^{(1)} = 0, \tag{2.5.24}$$

and

$$\mathbf{d}^{[-1]}(\sigma^{(0)}) = 0, \tag{2.5.25}$$

$$\mathbf{d}^{[0]}(\sigma^{(0)}) + \mathbf{d}^{[-1]}(\sigma^{(1)}) = 0. \tag{2.5.26}$$

Proof. We start with the trace identities. Since $\sigma^{(0)} = K^{(0)}$, (2.5.23) follows from (2.2.3) and $\mathrm{tr}_{\bar{g}_0}\bar{F}^{(1)} = 0$. Moreover, from (2.5.20) and (2.5.21) we have

$$\begin{aligned}
-\cos\left(\frac{u_0}{\lambda}\right)\mathrm{tr}_{\bar{F}^{(1)}}\sigma^{(0)} + \mathrm{tr}_{\bar{g}_0}\sigma^{(1)} &= -\cos\left(\frac{u_0}{\lambda}\right)\mathrm{tr}_{\bar{F}^{(1)}}K^{(0)} + \mathrm{tr}_{\bar{g}_0}K^{(1)} - \tau^{(1)} \\
&\quad - \mathrm{tr}_{\bar{g}_0}\mathbf{K}^{[-1]}(W^{(2)}) \\
&= -\cos\left(\frac{u_0}{\lambda}\right)\mathrm{tr}_{\bar{F}^{(1)}}K^{(0)} + \mathrm{tr}_{\bar{g}_0}K^{(1)} - \tau^{(1)} \\
&= 0
\end{aligned}$$

where we use (2.4.34) and (2.5.12). This proves (2.5.24).

We now look at the divergence identities. From (2.5.20), (2.5.2) and (2.4.24) we obtain

$$\mathbf{d}_\ell^{[-1]}(\sigma^{(0)}) = \frac{1}{2}|\nabla u_0|^2_{\bar{g}_0}\cos\left(\frac{u_0}{\lambda}\right)\bar{F}^{(1)}_{N_0\ell} = 0$$

which proves (2.5.25). We now compute the two parts of (2.5.26). Using (2.4.25) and (2.5.20) we obtain

$$\begin{aligned}
\mathbf{d}_\ell^{[0]}(\sigma^{(0)}) &= \mathrm{div}_{\bar{g}_0}K^{(0)}_\ell - \bar{g}_0^{ij}(\tilde{\Gamma}^{(0)})^a_{i\ell}K^{(0)}_{aj} \\
&= (\mathrm{div}_{\bar{g}_0}K_0)_\ell + \frac{1}{2}\sin\left(\frac{u_0}{\lambda}\right)\left(\mathrm{div}_{\bar{g}_0}|\nabla u_0|_{\bar{g}_0}\bar{F}^{(1)}\right)_\ell \\
&\quad - \bar{g}_0^{ij}(\tilde{\Gamma}^{(0)})^a_{i\ell}(K_0)_{aj} - \frac{1}{2}\sin\left(\frac{u_0}{\lambda}\right)|\nabla u_0|_{\bar{g}_0}\bar{g}_0^{ij}(\tilde{\Gamma}^{(0)})^a_{i\ell}\bar{F}^{(1)}_{aj}
\end{aligned}$$

We use (2.4.15), (2.5.5) and (2.2.12) to rewrite the second line in the previous expression and obtain

$$\begin{aligned}
\mathbf{d}_\ell^{[0]}(\sigma^{(0)}) &= (\mathrm{div}_{\bar{g}_0}K_0)_\ell + \frac{1}{2}\sin\left(\frac{u_0}{\lambda}\right)\left(\mathrm{div}_{\bar{g}_0}|\nabla u_0|_{\bar{g}_0}\bar{F}^{(1)}\right)_\ell \\
&\quad + \frac{1}{4}\sin\left(\frac{u_0}{\lambda}\right)\partial_\ell u_0\bar{F}^{(1)}_{ij}\partial_t\bar{g}_0^{ij} + 2\sin^2\left(\frac{u_0}{\lambda}\right)|\nabla u_0|_{\bar{g}_0}\partial_\ell u_0 F_0^2 \\
&= \frac{1}{2}\sin\left(\frac{u_0}{\lambda}\right)\left(\left(\mathrm{div}_{\bar{g}_0}|\nabla u_0|_{\bar{g}_0}\bar{F}^{(1)}\right)_\ell + \frac{1}{2}\partial_\ell u_0\bar{F}^{(1)}_{ij}\partial_t\bar{g}_0^{ij}\right) \qquad (2.5.27) \\
&\quad - \cos\left(\frac{2u_0}{\lambda}\right)|\nabla u_0|_{\bar{g}_0}\partial_\ell u_0 F_0^2
\end{aligned}$$

where we crucially use the background momentum constraint (2.2.2) to cancel the non-oscillating terms. We now look at $\mathbf{d}_\ell^{[-1]}(\sigma^{(1)})$. Using (2.4.24) and (2.5.21) we obtain

$$\begin{aligned}
\mathbf{d}_\ell^{[-1]}(\sigma^{(1)}) &= -|\nabla u_0|_{\bar{g}_0}\partial_\theta K^{(1)}_{N_0\ell} - \frac{1}{3}\partial_\ell u_0\partial_\theta\tau^{(1)} - \mathbf{M}_\ell^{[-2]}(W^{(2)}) \\
&= \partial_\theta\left(-|\nabla u_0|_{\bar{g}_0}K^{(1)}_{N_0\ell} - \partial_\ell u_0\tau^{(1)}\right)
\end{aligned}$$

where we use the equation satisfied by $W^{(2)}$ (see (2.5.16)). We use (2.5.11), (2.5.7) and (2.5.8) to obtain

$$\begin{aligned}
K^{(1)}_{N_0\ell} = -\cos\left(\frac{u_0}{\lambda}\right)&\left(\frac{\bar{F}^{(1)}_{k\ell}}{2}\left(\mathbf{D}_{N_0}N_0^k - \bar{g}_0^{kj}(K_0)_{N_0j}\right)\right. \\
&\left.+\frac{\bar{F}^{(1)}_{jk}}{4}\left(\bar{g}_0^{ik}\partial_i N_0^j + \frac{1}{2}(\partial_t - N_0)\bar{g}_0^{jk}\right)(N_0)_\ell + \frac{1}{2}|\nabla u_0|_{\bar{g}_0}\omega^{(2)}_{N_0\ell}\right) \\
+ \frac{3}{4}\sin\left(\frac{2u_0}{\lambda}\right)&|\nabla u_0|_{\bar{g}_0}F_0^2(N_0)_\ell
\end{aligned}$$

Using (2.5.12) and $\mathrm{tr}_{\bar{g}_0}\omega^{(2)} = 0$ we also obtain

$$\tau^{(1)} = -\frac{3}{4}\cos\left(\frac{u_0}{\lambda}\right)\bar{F}^{(1)}_{kj}\left(\bar{g}_0^{ij}\partial_i N_0^k + \frac{1}{2}\left(\partial_t - N_0\right)\bar{g}_0^{kj}\right) + \frac{1}{4}\sin\left(\frac{2u_0}{\lambda}\right)|\nabla u_0|_{\bar{g}_0}F_0^2. \quad (2.5.28)$$

This gives

$$\begin{aligned}
&-|\nabla u_0|_{\bar{g}_0}K^{(1)}_{N_0\ell} - \partial_\ell u_0\tau^{(1)} \\
&\quad = |\nabla u_0|_{\bar{g}_0}\cos\left(\frac{u_0}{\lambda}\right)\left(\frac{\bar{F}^{(1)}_{k\ell}}{2}\left(\mathbf{D}_{N_0}N_0^k - \bar{g}_0^{kj}(K_0)_{N_0j}\right)\right. \\
&\qquad\qquad \left. - \frac{\bar{F}^{(1)}_{jk}}{2}\left(\bar{g}_0^{ik}\partial_i N_0^j + \frac{1}{2}\left(\partial_t - N_0\right)\bar{g}_0^{jk}\right)(N_0)_\ell + \frac{1}{2}|\nabla u_0|_{\bar{g}_0}\omega^{(2)}_{N_0\ell}\right) \\
&\qquad + \frac{1}{2}\sin\left(\frac{2u_0}{\lambda}\right)|\nabla u_0|_{\bar{g}_0}F_0^2\partial_\ell u_0
\end{aligned}$$

Adding this to the expression of $\mathbf{d}^{[0]}_\ell(\sigma^{(0)})$ given by (2.5.27) we notice that the terms oscillating like 2θ cancel (see Remark 2.5.2 below) and we obtain

$$\begin{aligned}
&\mathbf{d}^{[0]}_\ell(\sigma^{(0)}) + \mathbf{d}^{[-1]}_\ell(\sigma^{(1)}) \\
&\quad = \sin\left(\frac{u_0}{\lambda}\right)\left[\frac{1}{2}\left(\mathrm{div}_{\bar{g}_0}|\nabla u_0|_{\bar{g}_0}\bar{F}^{(1)}\right)_\ell - \frac{|\nabla u_0|_{\bar{g}_0}}{2}\bar{F}^{(1)}_{k\ell}\left(\mathbf{D}_{N_0}N_0^k - \bar{g}_0^{kj}(K_0)_{N_0j}\right)\right. \\
&\qquad\qquad \left. - \frac{1}{2}|\nabla u_0|^2_{\bar{g}_0}\omega^{(2)}_{N_0\ell} - \frac{1}{2}\partial_\ell u_0\bar{F}^{(1)}_{kj}\bar{g}_0^{ij}\partial_i N_0^k + \frac{1}{4}\partial_\ell u_0\bar{F}^{(1)}_{kj}N_0\bar{g}_0^{kj}\right]
\end{aligned}$$

Using $\bar{F}^{(1)}_{N_0i} = 0$ we can compute the divergence of $|\nabla u_0|_{\bar{g}_0}\bar{F}^{(1)}$ and using in addition that $\omega^{(2)}_{N_0N_0} = 0$ we obtain

$$\mathbf{d}^{[0]}_{N_0}(\sigma^{(0)}) + \mathbf{d}^{[-1]}_{N_0}(\sigma^{(1)}) = 0.$$

The tangential components of $\mathbf{d}^{[0]}(\sigma^{(0)}) + \mathbf{d}^{[-1]}(\sigma^{(1)})$ are given by

$$\begin{aligned}
&\mathbf{d}^{[0]}_{\mathbf{j}}(\sigma^{(0)}) + \mathbf{d}^{[-1]}_{\mathbf{j}}(\sigma^{(1)}) \\
&\quad = \sin\left(\frac{u_0}{\lambda}\right)\left[\frac{1}{2}\left(\mathrm{div}_{\bar{g}_0}|\nabla u_0|_{\bar{g}_0}\bar{F}^{(1)}\right)_{\mathbf{j}}\right. \\
&\qquad\qquad \left. - \frac{|\nabla u_0|_{\bar{g}_0}}{2}\bar{F}^{(1)}_{k\mathbf{j}}\left(\mathbf{D}_{N_0}N_0^k - \bar{g}_0^{kj}(K_0)_{N_0j}\right) - \frac{1}{2}|\nabla u_0|^2_{\bar{g}_0}\omega^{(2)}_{N_0\mathbf{j}}\right].
\end{aligned}$$

This previous expression vanishes thanks to the choice of $\omega^{(2)}$ made in (2.4.7). This concludes the proof of (2.5.26).

□

Remark 2.5.2. *The cancellation of the* $\cos\left(\frac{2u_0}{\lambda}\right)$ *terms coming from* $\mathbf{d}^{[0]}_\ell(\sigma^{(0)})$ *and* $\mathbf{d}^{[-1]}_\ell(\sigma^{(1)})$ *in their sum seems to be linked to the weak polarized null condition satisfied by the semi-linear terms in the Einstein equations, which involved products of derivatives of the metric (see Section 3.2.1.3 in Chapter 3 for the definition of this condition). Indeed, these terms correspond to terms of the form* ΓK, $\Gamma\Gamma$ *or* KK *with* Γ *the Christoffel symbols and* K *the second fundamental form, i.e terms of the form* $\partial g \partial g$.

2.5.4 An exact TT-tensor

In the next section, we are going to solve completely the constraint equations, i.e solve for the remainders in the high-frequency ansatz (2.3.9)-(2.3.10). We will thus need the full expression of the parameters of the conformal method. While γ and τ are already fully defined, σ is only partially known yet and is only an *almost* TT-tensor, as it was shown in the previous section. In this section we finish the construction of σ. We choose the following ansatz

$$\sigma = \sigma^{(0)} + \lambda\sigma^{(1)} + \lambda^2\left(\sigma^{(2)} + L_\gamma Y\right) + \frac{\lambda^3}{3}\mathfrak{f}\gamma. \tag{2.5.29}$$

In this expression, $\sigma^{(0)}$ and $\sigma^{(1)}$ are given by (2.5.20) and (2.5.21) and $\sigma^{(2)}$, Y and $\mathfrak{f}$ are yet to be defined such that

$$\mathrm{tr}_\gamma\sigma = 0 \quad \text{and} \quad \mathrm{div}_\gamma\sigma = 0. \tag{2.5.30}$$

Let us explain the ansatz (2.5.29). Since we need to satisfy the compatibility with the space-time ansatz (2.5.19), we can't modifiy the order λ^0 and λ^1 of σ. Thus, a non-oscillating remainder can only appear at the order λ^2. However, such a remainder would not be able to solve the λ^1 level of $\mathrm{div}_\gamma\sigma = 0$ (recall (2.5.22)). Therefore we need to add an oscillating field at the order λ^2, i.e $\sigma^{(2)}$. This field will also be able to solve the λ^2 level of $\mathrm{tr}_\gamma\sigma = 0$. Finally the remainder is chosen of the form $L_\gamma Y + \frac{\lambda}{3}\mathfrak{f}\gamma$, where the vector field Y ensures $\mathrm{div}_\gamma\sigma = 0$ and the scalar function $\mathfrak{f}$ ensures $\mathrm{tr}_\gamma\sigma = 0$.

We now derive the equations for $\sigma^{(2)}$, Y and $\mathfrak{f}$, which illustrates the above discussion. Thanks to Lemma 2.5.3 the equations (2.5.30) rewrite as

$$\lambda^2\left(\mathrm{tr}_{\gamma^{(\geq 2)}}\sigma^{(0)} + \mathrm{tr}_{\gamma^{(\geq 1)}}\sigma^{(1)} + \mathrm{tr}_\gamma\sigma^{(2)}\right) + \lambda^3\mathfrak{f} = 0$$

and

$$\begin{aligned}\lambda&\left(\mathbf{d}^{[1]}(\sigma^{(0)}) + \mathbf{d}^{[0]}(\sigma^{(1)}) + \mathbf{d}^{[-1]}(\sigma^{(2)}) + M^{[-1]}(Y)\right)\\ &+\lambda^2\left(\mathbf{d}^{[\geq 2]}(\sigma^{(0)}) + \mathbf{d}^{[\geq 1]}(\sigma^{(1)}) + \mathbf{d}^{[\geq 0]}(\sigma^{(2)}) + M^{[\geq 0]}(Y) + \frac{\lambda}{3}\mathrm{d}\mathfrak{f}\right) = 0\end{aligned}$$

where $\mathrm{d}\mathfrak{f}$ also includes derivatives of the oscillating parts of $\mathfrak{f}$, which implies in particular that $\mathrm{d}\mathfrak{f} = O\left(\lambda^{-1}\right)$. In order to solve these two equations, we want $\sigma^{(2)}$, Y and $\mathfrak{f}$ to satisfy the following coupled system (recall the expression of $\mathbf{d}^{[-1]}$ given by (2.4.24)):

$$\mathrm{tr}_{\bar{g}_0}\sigma^{(2)} = -\mathrm{tr}_{\gamma^{(2)}}\sigma^{(0)} - \mathrm{tr}_{\gamma^{(1)}}\sigma^{(1)} \tag{2.5.31}$$

$$\mathfrak{f} = -\mathrm{tr}_{\gamma^{(\geq 1)}}\sigma^{(2)} - \mathrm{tr}_{\gamma^{(\geq 2)}}\sigma^{(1)} - \mathrm{tr}_{\gamma^{(\geq 3)}}\sigma^{(0)} \tag{2.5.32}$$

$$-|\nabla u_0|_{\bar{g}_0}\partial_\theta\sigma^{(2)}_{N_0\ell} = -M_\ell^{[-1]}(Y) - \mathbf{d}_\ell^{[1]}(\sigma^{(0)}) - \mathbf{d}_\ell^{[0]}(\sigma^{(1)}) \tag{2.5.33}$$

$$M^{[\geq 0]}(Y) = -\mathbf{d}^{[\geq 2]}(\sigma^{(0)}) - \mathbf{d}^{[\geq 1]}(\sigma^{(1)}) - \mathbf{d}^{[\geq 0]}(\sigma^{(2)}) - \frac{\lambda}{3}\mathrm{d}\mathfrak{f} \tag{2.5.34}$$

Equations (2.5.31) and (2.5.32) ensure $\mathrm{tr}_\gamma\sigma = 0$ while (2.5.33) and (2.5.34) ensure $\mathrm{div}_\gamma\sigma = 0$. The rest of this section is devoted to the resolution of the system (2.5.31)-(2.5.32)-(2.5.33)-(2.5.34). It presents a triangular structure, despite the term $M^{[-1]}(Y)$ in (2.5.33).

2.5.4.1 Definition of $\sigma^{(2)}$ and $\mathfrak{f}$

We start by solving the non-differential equations of the previous system, that is (2.5.31)-(2.5.32)-(2.5.33). The first step is to show that the RHS of (2.5.33) is purely oscillating, which is a necessary condition since the LHS is a ∂_θ derivative. Thanks to (2.4.37), the first term in the RHS is oscillating, and the next lemma deals with the last two.

Lemma 2.5.4. *The following oscillating behaviour holds*

$$\mathbf{d}^{[1]}(\sigma^{(0)}) + \mathbf{d}^{[0]}(\sigma^{(1)}) \overset{\text{osc}}{\sim} \cos(\theta) + \sin(2\theta) + \cos(3\theta).$$

Proof. From (2.4.27) we have

$$\mathbf{d}^{[0]}(\sigma^{(1)}) \overset{\text{osc}}{\sim} (1 + \sin(\theta))\sigma^{(1)}$$

and from (2.5.21) we have

$$\begin{aligned} \sigma^{(1)} &\overset{\text{osc}}{\sim} K^{(1)} + \cos(\theta) + \tau^{(1)} + \mathbf{K}^{[-1]}(W^{(2)}) \\ &\overset{\text{osc}}{\sim} \cos(\theta) + \sin(2\theta) + \partial_\theta W^{(2)} \end{aligned}$$

where we used (2.5.11) and (2.5.12). Now using (2.5.18) we conclude that

$$\mathbf{d}^{[0]}(\sigma^{(1)}) \overset{\text{osc}}{\sim} \cos(\theta) + \sin(2\theta).$$

Now, from (2.4.28) we have

$$\mathbf{d}^{[1]}(\sigma^{(0)}) \overset{\text{osc}}{\sim} \sin(\theta)\partial_\theta\sigma^{(0)} + (\cos(\theta) + \sin(2\theta))\sigma^{(0)}$$

and from (2.5.20) and (2.5.2) we have

$$\sigma^{(0)} \overset{\text{osc}}{\sim} 1 + \sin(\theta).$$

This concludes the proof of the lemma. □

We have shown that the RHS of (2.5.33) is purely oscillating, which allows us to formally integrate this equation in θ and obtain $\sigma^{(2)}$. More precisely, thanks to $\bar{F}^{(1)}_{iN_0} = 0$, (2.4.37) implies that $M^{[-1]}_{N_0}(Y) = 0$. Therefore, (2.5.33) gives us $\sigma^{(2)}_{N_0 N_0}$ as a function of lower order terms in the construction, i.e $\sigma^{(0)}$ and $\sigma^{(1)}$. Then, (2.5.31) gives us $\sigma^{(2)}_{\mathbf{11}}$ and $\sigma^{(2)}_{\mathbf{22}}$ as a function of $\sigma^{(2)}_{N_0 N_0}$, $\sigma^{(0)}$ and $\sigma^{(1)}$. All together, the diagonal components of $\sigma^{(2)}$ in the frame $(N_0, e_\mathbf{1}, e_\mathbf{2})$ are functions of $\sigma^{(0)}$ and $\sigma^{(1)}$ satisfying

$$\left|\sigma^{(2)}_{N_0N_0}\right| + \left|\sigma^{(2)}_{\mathbf{11}}\right| + \left|\sigma^{(2)}_{\mathbf{22}}\right| \lesssim \left|\mathbf{d}^{[1]}(\sigma^{(0)})\right| + \left|\mathbf{d}^{[0]}(\sigma^{(1)})\right| + \left|(\gamma^{-1})^{(2)}\sigma^{(0)}\right| + \left|(\gamma^{-1})^{(1)}\sigma^{(1)}\right| \quad (2.5.35)$$

with a high-frequency behaviour, meaning that we lose one power of λ for each derivatives.

Remark 2.5.3. *Note that we don't impose conditions on* $\sigma^{(2)}_{\mathbf{11}}$ *and* $\sigma^{(2)}_{\mathbf{22}}$ *separately but only on their sum.*

The other components of $\sigma^{(2)}$ in the frame $(N_0, e_\mathbf{1}, e_\mathbf{2})$ as well as the scalar function $\mathfrak{f}$ depends on the vector field Y, which is yet to be defined. The equation (2.5.33) allows us to define $\sigma^{(2)}_{N_0\mathbf{i}}$ as a linear function of Y and of $\sigma^{(0)}$ and $\sigma^{(1)}$. More precisely we obtain

$$\left|\sigma^{(2)}_{N_0\mathbf{1}}(Y)\right| + \left|\sigma^{(2)}_{N_0\mathbf{2}}(Y)\right| \lesssim \left|\bar{F}^{(1)}Y\right| + \left|\mathbf{d}^{[1]}(\sigma^{(0)})\right| + \left|\mathbf{d}^{[0]}(\sigma^{(1)})\right| \quad (2.5.36)$$

with a high-frequency behaviour. Since this component doesn't appear in the equations $\sigma^{(2)}$ needs to solve, we set $\sigma^{(2)}_{\mathbf{12}} = 0$.

The scalar function $\mathfrak{f}$ is actually already defined by (2.5.32), but as $\sigma^{(2)}$ is a function of Y, so is $\mathfrak{f}$. Therefore, thanks to (2.5.35) and (2.5.36) $\mathfrak{f}$ satisfies

$$\begin{aligned}|\mathfrak{f}(Y)| &\lesssim \left|(\gamma^{-1})^{(\geq 1)}\sigma^{(2)}\right| + \left|(\gamma^{-1})^{(\geq 2)}\sigma^{(1)}\right| + \left|(\gamma^{-1})^{(\geq 3)}\sigma^{(0)}\right| \\ &\lesssim \left|(\gamma^{-1})^{(\geq 1)}\bar{F}^{(1)}Y\right| + \left|(\gamma^{-1})^{(\geq 1)}\mathbf{d}^{[1]}(\sigma^{(0)})\right| + \left|(\gamma^{-1})^{(\geq 1)}\mathbf{d}^{[0]}(\sigma^{(1)})\right| \\ &\quad + \left|(\gamma^{-1})^{(\geq 1)}(\gamma^{-1})^{(2)}\sigma^{(0)}\right| + \left|(\gamma^{-1})^{(\geq 1)}(\gamma^{-1})^{(1)}\sigma^{(1)}\right| \\ &\quad + \left|(\gamma^{-1})^{(\geq 2)}\sigma^{(1)}\right| + \left|(\gamma^{-1})^{(\geq 3)}\sigma^{(0)}\right| \end{aligned} \tag{2.5.37}$$

with a high-frequency behaviour.

2.5.4.2 Solving for Y

To conclude the construction of σ, it remains to solve (2.5.34). As explained after Lemma 2.4.6, this is done by actually "replacing" the operator $M^{[\geq 0]}$ by $\mathrm{div}_e L_e$ together with a fixed point argument. More precisely, we define a map $\mathbf{\Psi}$

$$\mathbf{\Psi} : \mathcal{B} \longrightarrow \mathcal{B}$$

such that $\mathbf{\Psi}(Y)$ is the solution of

$$\begin{aligned}\mathrm{div}_e L_e \mathbf{\Psi}(Y) = \mathrm{div}_e L_e Y - M^{[\geq 0]}(Y) - \mathbf{d}^{[\geq 2]}(\sigma^{(0)}) \\ - \mathbf{d}^{[\geq 1]}(\sigma^{(1)}) - \mathbf{d}^{[\geq 0]}(\sigma^{(2)}(Y)) - \frac{\lambda}{3}\mathrm{d}\mathfrak{f}(Y)\end{aligned} \tag{2.5.38}$$

and where

$$\mathcal{B} = \left\{ Z \in H^2_\delta \;\middle|\; \|Z\|_{H^2_\delta} \leq C_1\varepsilon \right\}. \tag{2.5.39}$$

with $C_1 > 0$ to be chosen later. Note that any fixed point of $\mathbf{\Psi}$ is a solution of (2.5.34). In order to prove the existence of a fixed point, we need to show that $\mathbf{\Psi}$ is well-defined and is a contraction.

Proposition 2.5.1. *If C_1 is large enough and ε is small enough, then the map $\mathbf{\Psi}$ is well-defined and is a contraction.*

Proof. Let $Y \in \mathcal{B}$. We bound the RHS of (2.5.38) in $L^2_{\delta+2}$:

$$\begin{aligned}\|\text{RHS of (2.5.38)}\|_{L^2_{\delta+2}} &\lesssim \left\|\mathrm{div}_e L_e Y - M^{[\geq 0]}(Y)\right\|_{L^2_{\delta+2}} + \left\|\mathbf{d}^{[\geq 0]}(\sigma^{(2)}(Y))\right\|_{L^2_{\delta+2}} + \lambda\left\|\mathrm{d}[\mathfrak{f}(Y)]\right\|_{L^2_{\delta+2}} \\ &\quad + \left\|\mathbf{d}^{[\geq 2]}(\sigma^{(0)}) + \mathbf{d}^{[\geq 1]}(\sigma^{(1)})\right\|_{L^2} \\ &=: A + B + C + D\end{aligned}$$

where we omitted the weights for the last term since it is compactly supported. For A, (2.4.38) gives

$$A \lesssim \left\|(\gamma^{-1} - e^{-1})\partial^2 Y\right\|_{L^2_{\delta+2}} + \|\partial\gamma\partial Y\|_{L^2_{\delta+2}} + \left\|(\partial\gamma)^2 Y\right\|_{L^2_{\delta+2}} + \left\|(\partial^2\gamma)^{(\geq 0)} Y\right\|_{L^2_{\delta+2}}$$

where we also used the fact that the coefficients of γ are bounded. We bound all the metric terms in L^∞ using the background regularity (2.2.4) and (2.4.9). More precisely, we have

$$\left\|\gamma^{-1} - e^{-1}\right\|_{L^\infty} \lesssim \|\bar{g}_0 - e\|_{L^\infty} + \lambda \left\|\bar{F}^{(1)}\right\|_{L^\infty} + \lambda^2 \left\|\omega^{(2)}\right\|_{L^\infty}, \tag{2.5.40}$$

$$\begin{aligned} \|\partial\gamma\|_{L^\infty} &\lesssim \|\partial\bar{g}_0\|_{L^\infty} + \left\|\bar{F}^{(1)}\right\|_{L^\infty} \\ &\quad + \lambda\left(\left\|\partial\bar{F}^{(1)}\right\|_{L^\infty} + \left\|\omega^{(2)}\right\|_{L^\infty}\right) + \lambda^2\left\|\partial\omega^{(2)}\right\|_{L^\infty}, \end{aligned} \tag{2.5.41}$$

$$\begin{aligned} \left\|(\partial^2\gamma)^{(\geq 0)}\right\|_{L^\infty} &\lesssim \|\partial^2\bar{g}_0\|_{L^\infty} + \left\|\partial\bar{F}^{(1)}\right\|_{L^\infty} + \left\|\omega^{(2)}\right\|_{L^\infty} \\ &\quad + \lambda\left(\left\|\partial^2\bar{F}^{(1)}\right\|_{L^\infty} + \left\|\partial\omega^{(2)}\right\|_{L^\infty}\right) + \lambda^2\left\|\partial^2\omega^{(2)}\right\|_{L^\infty}. \end{aligned} \tag{2.5.42}$$

Using (2.2.4), (2.4.9), (2.4.7) and $\|Y\|_{H^2_\delta} \leq C_1\varepsilon$, the estimates (2.5.40)-(2.5.41)-(2.5.42) then imply that $A \lesssim C_1\varepsilon^2$.

The term D depends only on previous terms of the construction so (2.4.26) simply gives $D \lesssim \varepsilon$. The maps $Y \longmapsto \sigma^{(2)}(Y)$ and $Y \longmapsto \mathfrak{f}(Y)$ are affine with coefficients as D (see (2.5.35), (2.5.36) and (2.5.37)) so a combination of the two previous arguments gives $B+C \lesssim C_1\varepsilon^2+\varepsilon$. Note that for C we need to compensate the loss in power of λ when differenciating the oscillating parts of $\mathfrak{f}$ with the λ in front.

We have proved that $A + B + C + D \lesssim C_1\varepsilon^2 + \varepsilon$. In particular, this allows us to use the second part of Proposition 2.1.2 to prove that there exists a unique $\mathbf{\Psi}(Y) \in H^2_\delta$ solving (2.5.38). Moreover we have

$$\|\mathbf{\Psi}(Y)\|_{H^2_\delta} \lesssim C_1\varepsilon^2 + \varepsilon.$$

Therefore, taking C_1 large compared to the numerical constant appearing in these estimates and ε small compared to 1 proves that $\mathbf{\Psi}$ is well-defined and maps $\mathcal{B}$ to itself.

To prove that $\mathbf{\Psi}$ is a contraction, we consider Y_a and Y_b two elements of $\mathcal{B}$. By substracting the equations satisfied by $\mathbf{\Psi}(Y_a)$ and $\mathbf{\Psi}(Y_b)$ we obtain the equation for their difference

$$\begin{aligned} \mathrm{div}_e L_e\left(\mathbf{\Psi}(Y_a) - \mathbf{\Psi}(Y_b)\right) &= \left(\mathrm{div}_e L_e - M^{[\geq 0]}\right)(Y_a - Y_b) \\ &\quad - \mathbf{d}^{[\geq 0]}\left(\sigma^{(2)}(Y_a) - \sigma^{(2)}(Y_b)\right) - \frac{\lambda}{3}\mathrm{d}\left(\mathfrak{f}(Y_a) - \mathfrak{f}(Y_b)\right). \end{aligned} \tag{2.5.43}$$

Using again the fact that $Y \longmapsto \sigma^{(2)}(Y)$ and $Y \longmapsto \mathfrak{f}(Y)$ are affine and using (2.4.38) for $\mathrm{div}_e L_e - M^{[\geq 0]}$, we can prove that

$$\|\text{RHS of (2.5.43)}\|_{L^2_{\delta+2}} \lesssim \varepsilon \|Y_a - Y_b\|_{H^2_\delta}.$$

Therefore, taking ε small enough ensures that $\mathbf{\Psi}$ is a contraction. □

Thanks to this proposition, the Banach fixed point theorem implies the existence of $Y \in \mathcal{B}$ solving (2.5.34), and therefore $(\sigma^{(2)}(Y), \mathfrak{f}(Y), Y)$ solves (2.5.31)-(2.5.32)-(2.5.33)-(2.5.34).

We can also prove that Y enjoys higher regularity. Indeed we can bound the RHS of (2.5.38) in higher order Sobolev spaces and use elliptic estimates for $\mathrm{div}_e L_e$ as in the previous proposition. The worse term in (2.5.38) is given by $\nabla\sigma^{(1)}$ which is bounded in H^{N-2} (see

(2.5.21), (2.5.11) and (2.5.9)) and in terms of decay the worse term is $\nabla\sigma^{(0)}$ (see (2.5.20), (2.5.2) and (2.2.4)). Therefore we obtain a solution Y of (2.5.34) such that $Y \in H^N_\delta$ and

$$\|Y\|_{H^{k+2}_\delta} \lesssim \frac{\varepsilon}{\lambda^k} \tag{2.5.44}$$

for $k \in [\![0, N-2]\!]$, where the loss of λ powers is due to the high-frequency character of each term in (2.5.38). We summarize what we know on the parameter σ in the following corollary.

Corollary 2.5.1. *The tensor σ defined by (2.5.29) is a TT-tensor for the metric γ, belongs to $H^{N-1}_{\delta+1}$ and satisfies*

$$\max_{k\in[\![0,N-3]\!]} \lambda^{k+2} \|\sigma\|_{H^{k+2}_{\delta+1}} + \max_{k\in[\![0,N-3]\!]} \lambda^k \left\|\nabla^k\sigma\right\|_{L^\infty} \lesssim \varepsilon. \tag{2.5.45}$$

Proof. The oscillating terms $\sigma^{(0)}$ and $\sigma^{(1)}$ lose one λ power for each derivatives and we can estimate the actual tensors by estimating (2.5.20)-(2.5.21) directly in weighted Sobolev spaces or in L^∞ using Sobolev embeddings of Proposition 2.1.1. Moreover, we can neglect $\sigma^{(2)}(Y)$ and $\mathfrak{f}(Y)$ and focus on $L_\gamma Y$ in (2.5.29) which rewrites broadly as a ∇Y term since we can put the γ and $\partial\gamma$ term in L^∞. Therefore, the estimate (2.5.45) follows directly from (2.5.44) and Sobolev embeddings. □

2.6 Exact solution to the constraint equations

We are now ready to solve the constraint equations (2.3.3) and (2.3.4). The parameters of these equations are γ, τ and σ. The metric γ and the scalar function τ are fully known thanks to Section 2.4.1 and (2.5.12). The TT-tensor σ has been defined in Sections 2.5.4 and 2.5.3. Recall that the solutions of (2.3.3)-(2.3.4) are of the form

$$W = \lambda^2\left(W^{(2)}\left(\frac{u_0}{\lambda}\right) + \widetilde{W}\right) + \lambda^3 W^{(3)}\left(\frac{u_0}{\lambda}\right),$$
$$\varphi = 1 + \lambda^2\left(\varphi^{(2)}\left(\frac{u_0}{\lambda}\right) + \widetilde{\varphi}\right) + \lambda^3\varphi^{(3)}\left(\frac{u_0}{\lambda}\right),$$

where $\varphi^{(2)}$, $\varphi^{(3)}$ and $W^{(2)}$ are defined in Sections 2.5.1.1, 2.5.1.2 and 2.5.2 respectively. Therefore, it remains to construct $\widetilde{\varphi}$, $\widetilde{W}$ and $W^{(3)}$.

2.6.1 System for the remainders

The construction of Sections 2.5.1.1, 2.5.1.2 and 2.5.2 ensures that the constraint equations (2.3.3) and (2.3.4) are partly solved. More precisely, it remains to solve the $\lambda^{\geq 2}$ levels of (2.3.3) and the $\lambda^{\geq 1}$ levels of (2.3.4). In this section, we compute the exact equations this gives for $\widetilde{\varphi}$, $\widetilde{W}$ and $W^{(3)}$.

2.6.1.1 Definition of $W^{(3)}(\widetilde{W})$

The purpose of the oscillating vector field $W^{(3)}$ is to solve the λ^1 momentum level. However, since the conformal Laplacian $\mathrm{div}_\gamma L_\gamma$ loses one power of λ even when applied to a non-oscillating field such as $\widetilde{W}$ (see Lemma 2.4.6), the latter is a source term in the equation for $W^{(3)}$. This explains why $W^{(3)}$ is considered as part of the remainders, when $\varphi^{(3)}$ is not.

We define $W^{(3)}$ as a function of $\widetilde{W}$. The λ^1 momentum level writes

$$\mathbf{M}^{[-2]}_\ell(W^{(3)}) + M^{[-1]}_\ell(\widetilde{W}) + \mathbf{M}^{[-1]}_\ell(W^{(2)}) = \frac{2}{3}\partial_\ell\tau^{(1)}. \tag{2.6.1}$$

Thanks to (2.4.43) and (2.4.37) this is equivalent to

$$\partial_\theta^2 W_\ell^{(3)} + \frac{1}{3}(N_0)_\ell \partial_\theta^2 W_{N_0}^{(3)} = -\cos\left(\frac{u_0}{\lambda}\right) \bar{g}_0^{ij} \bar{F}_{i\ell}^{(1)} \widetilde{W}_j + \frac{1}{|\nabla u_0|^2_{\bar{g}_0}} \left(\frac{2}{3}\partial_\ell \tau^{(1)} - \mathbf{M}_\ell^{[-1]}(W^{(2)}) \right).$$

Lemma 2.4.8 then gives

$$\begin{aligned} \partial_\theta^2 W_\ell^{(3)} &= -\cos\left(\frac{u_0}{\lambda}\right) \bar{g}_0^{ij} \bar{F}_{i\ell}^{(1)} \widetilde{W}_j \\ &\quad + \frac{1}{|\nabla u_0|^2_{\bar{g}_0}} \left[\frac{2}{3}\partial_\ell \tau^{(1)} - \mathbf{M}_\ell^{[-1]}(W^{(2)}) - \frac{1}{4}(N_0)_\ell \left(\frac{2}{3} N_0 \tau^{(1)} - \mathbf{M}_{N_0}^{[-1]}(W^{(2)}) \right) \right]. \end{aligned} \tag{2.6.2}$$

Let us check that the RHS of this equation is purely oscillating. Since $\tau^{(1)}$ is purely oscillating (see (2.5.28)), we only need to check $\mathbf{M}_\ell^{[-1]}(W^{(2)})$. For this we use first (2.4.42) and then (2.5.18), this gives

$$\begin{aligned} \mathbf{M}^{[-1]}(W^{(2)}) &\overset{\text{osc}}{\sim} \cos(\theta) W^{(2)} + \cos(\theta)\partial_\theta^2 W^{(2)} + (1+\sin(\theta))\partial_\theta W^{(2)} \\ &\overset{\text{osc}}{\sim} \cos(\theta) + \sin(2\theta) + \cos(3\theta). \end{aligned}$$

The RHS of (2.6.2) is thus purely oscillating and we can integrate it twice with respect to θ. We obtain

$$W_\ell^{(3)}(\widetilde{W}) = \cos\left(\frac{u_0}{\lambda}\right) \bar{g}_0^{ij} \bar{F}_{i\ell}^{(1)} \widetilde{W}_j + W_\ell^{(3,rest)} \tag{2.6.3}$$

where $W^{(3,rest)}$ satisfies

$$\left| W^{(3,rest)} \right| \lesssim \left| \partial \tau^{(1)} \right| + \left| \mathbf{M}^{[-1]}(W^{(2)}) \right| \tag{2.6.4}$$

with a high-frequency behaviour.

2.6.1.2 The system for $\widetilde{W}$ and $\widetilde{\varphi}$

In this section, we will expand in the most concise way the non-linearities involved in the equations for $\widetilde{\varphi}$ and $\widetilde{W}$. We start with the equation for $\widetilde{W}$, which, if we drop the vectorial notation, writes

$$M^{[\geq 0]}(\widetilde{W}) = -\mathbf{M}^{[\geq -1]}(W^{(3)}(\widetilde{W})) - \mathbf{M}^{[\geq 0]}(W^{(2)}) + \frac{2}{3}(\varphi^6 \partial \tau)^{(\geq 2)} \tag{2.6.5}$$

The following lemma expands the non-linearity in (2.6.5).

Lemma 2.6.1. *We have*

$$\frac{2}{3}(\varphi^6 \partial_\ell \tau)^{(\geq 2)} = \mathbf{a}_0 + \sum_{k=1}^{6} \lambda^{2(k-1)} \mathbf{a}_k \widetilde{\varphi}^k \tag{2.6.6}$$

where for $k \in [\![0,6]\!]$, $\mathbf{a}_k$ is supported in B_R and

$$\max_{i \in [\![0,N-5]\!]} \lambda^i \left\| \nabla^i \mathbf{a}_k \right\|_{L^\infty} \lesssim \varepsilon. \tag{2.6.7}$$

Proof. Recall that $\tau = \lambda\tau^{(1)}$ implies $\partial\tau = O\left(\lambda^0\right)$, thus we only need to expand

$$\left(1 + \lambda^2\left(\varphi^{(2)} + \widetilde{\varphi}\right) + \lambda^3\varphi^{(3)}\right)^6$$

and only keep the terms of order λ^2 or more, which only excludes the term 1. The coefficient $\mathbf{a}_0$ in (2.6.6) contains all the terms where $\widetilde{\varphi}$ doesn't appear, it is thus a polynomial in $\varphi^{(2)}$ and $\varphi^{(3)}$ with no constant term and multiplied by $\partial\tau$. Therefore, $\mathbf{a}_0$ shares the same support property as $\varphi^{(2)}$ and $\varphi^{(3)}$ and the estimate (2.6.7) follows from (2.5.6), (2.5.15) and (2.5.12). If $k \in [\![0,6]\!]$, the same reasoning applies but $\mathbf{a}_k$ is now a polynomial in $\varphi^{(2)}$ and $\varphi^{(3)}$ with a constant term. But as this polynomial is still multiplied by $\partial\tau$, the support property and the estimate still hold. □

The equation for $\widetilde{\varphi}$ writes

$$\begin{aligned} 8\Delta_\gamma\widetilde{\varphi} = -8\sum_{i=2}^{3}\mathbf{H}^{[\geq 2-i]}(\varphi^{(i)}) + R^{(\geq 2)} + R(\gamma)\left(\varphi^{(2)} + \widetilde{\varphi} + \lambda\varphi^{(3)}\right) \\ + \frac{2}{3}(\tau^2\varphi^5)^{(\geq 2)} - \left(|\sigma + L_\gamma W|^2_\gamma\,\varphi^{-7}\right)^{(\geq 2)}. \end{aligned} \tag{2.6.8}$$

The next two lemmas expand the non-linearities in (2.6.8).

Lemma 2.6.2. *We have*

$$\frac{2}{3}(\tau^2\varphi^5)^{(\geq 2)} = \mathbf{b}_0 + \sum_{k=1}^{5}\lambda^{2k}\mathbf{b}_k\widetilde{\varphi}^k$$

where for $k \in [\![0,5]\!]$, $\mathbf{b}_k$ *is supported in* B_R *and*

$$\max_{i\in[\![0,N-5]\!]}\lambda^i\left\|\nabla^i\mathbf{b}_k\right\|_{L^\infty} \lesssim \varepsilon. \tag{2.6.9}$$

The proof of Lemma 2.6.2 is left to the reader since it is very similar to the one of Lemma 2.6.1. We now expand the non-linearities with a negative power of φ.

Lemma 2.6.3. *There exists a universal constant* $C_{emb} > 0$ *such that if* $\|\widetilde{\varphi}\|_{H^2_\sigma} < C_{emb}^{-1}$ *and if* ε *is small enough, then we have*

$$\left(|\sigma + L_\gamma W|^2_\gamma\,\varphi^{-7}\right)^{(\geq 2)} = \left(|\sigma + L_\gamma W|^2_\gamma\right)^{(\geq 2)} + |\sigma + L_\gamma W|^2_\gamma\left(\mathbf{c}_0 + \sum_{k\geq 1}\mathbf{c}_k\lambda^{2(k-1)}\widetilde{\varphi}^k\right)$$

where the $\mathbf{c}_k$ *satisfy:*

- $\mathbf{c}_0$ *is supported in* B_R *and we have*

$$\max_{i\in[\![0,N-5]\!]}\lambda^i\left\|\nabla^i\mathbf{c}_0\right\|_{L^\infty} \lesssim \varepsilon, \tag{2.6.10}$$

- *if* $k \geq 1$, *we have*

$$\max_{i\in[\![0,N-5]\!]}\lambda^i\left\|\nabla^i\mathbf{c}_k\right\|_{L^\infty} \lesssim 1. \tag{2.6.11}$$

Proof. The constant C_{emb} is the one appearing in the embedding $H^2_\delta \hookrightarrow L^\infty$ (see Proposition 2.1.1), i.e

$$\|u\|_{L^\infty} \leq C_{emb}\|u\|_{H^2_\delta}$$

for all $u \in H^2_\delta$. Now if ε is small enough and if $\|\widetilde{\varphi}\|_{H^2_\delta} < C_{emb}^{-1}$, we have

$$\left\|\varphi^{(2)} + \widetilde{\varphi} + \lambda\varphi^{(3)}\right\|_{L^\infty} \leq 1.$$

This allows us to expand $\varphi^{-7} = \left(1 + \lambda^2\left(\varphi^{(2)} + \widetilde{\varphi} + \lambda\varphi^{(3)}\right)\right)^{-7}$. Indeed there exists a sequence $(\mathbf{c}_k)_{k\in\mathbb{N}}$ such that

$$\varphi^{-7} = 1 + \lambda^2\left(\mathbf{c}_0 + \sum_{k\geq 1}\mathbf{c}_k\lambda^{2(k-1)}\widetilde{\varphi}^k\right)$$

where $\mathbf{c}_0$ is a polynomial in $\varphi^{(2)}$ and $\varphi^{(3)}$ with no constant term and $\mathbf{c}_k$ for $k \geq 1$ is a polynomial in $\varphi^{(2)}$ and $\varphi^{(3)}$ with a constant term bounded but not compactly supported. This justifies the estimates (2.6.10) and (2.6.11). Therefore, we have

$$\left(|\sigma + L_\gamma W|^2_\gamma\,\varphi^{-7}\right)^{(\geq 2)} = \left(|\sigma + L_\gamma W|^2_\gamma\right)^{(\geq 2)} + |\sigma + L_\gamma W|^2_\gamma\left(\mathbf{c}_0 + \sum_{k\geq 1}\mathbf{c}_k\lambda^{2(k-1)}\widetilde{\varphi}^k\right)$$

which concludes the proof. □

Putting Lemmas 2.6.1, 2.6.2 and 2.6.3 together, we obtain the final form of the system solved by $\widetilde{\varphi}$ and $\widetilde{W}$:

$$M^{[\geq 0]}(\widetilde{W}) = -\mathbf{M}^{[\geq -1]}(W^{(3)}(\widetilde{W})) + \sum_{k=1}^{6}\lambda^{2(k-1)}\mathbf{a}_k\widetilde{\varphi}^k + \mathcal{R}_{\mathrm{mom}}, \tag{2.6.12}$$

$$8\Delta_\gamma\widetilde{\varphi} = R(\gamma)\widetilde{\varphi} + \sum_{k=1}^{5}\lambda^{2k}\mathbf{b}_k\widetilde{\varphi}^k \tag{2.6.13}$$
$$- \left(|\sigma + L_\gamma W|^2_\gamma\right)^{(\geq 2)} - |\sigma + L_\gamma W|^2_\gamma\left(\mathbf{c}_0 + \sum_{k\geq 1}\mathbf{c}_k\lambda^{2(k-1)}\widetilde{\varphi}^k\right) + \mathcal{R}_{\mathrm{Ham}},$$

where we define the following remainders

$$\mathcal{R}_{\mathrm{mom}} = -\mathbf{M}^{[\geq 0]}(W^{(2)}) + \mathbf{a}_0, \tag{2.6.14}$$

$$\mathcal{R}_{\mathrm{Ham}} = -8\sum_{i=2}^{3}\mathbf{H}^{[\geq 2-i]}(\varphi^{(i)}) + R^{(\geq 2)} + R(\gamma)\left(\varphi^{(2)} + \lambda\varphi^{(3)}\right) + \mathbf{b}_0. \tag{2.6.15}$$

2.6.2 Fixed point argument

In this section, we solve (2.6.12) and (2.6.13) by a fixed point argument. As in Section 2.5.4.2, the idea is to replace the operators depending on γ by their Euclidean equivalent and use the smallness of $\gamma - e$. We introduce the map $\mathbf{\Phi}$

$$\mathbf{\Phi} : \mathcal{B}\times\mathcal{B} \longrightarrow \mathcal{B}\times\mathcal{B}$$
$$(\widetilde{\varphi}, \widetilde{W}) \longmapsto \left(\mathbf{\Phi}_1(\widetilde{\varphi}), \mathbf{\Phi}_2(\widetilde{W})\right)$$

such that $\mathbf{\Phi}_1(\widetilde{\varphi})$ and $\mathbf{\Phi}_2(\widetilde{W})$ are solutions of the coupled system

$$\begin{aligned} \mathrm{div}_e L_e \mathbf{\Phi}_2(\widetilde{W}) &= \mathrm{div}_e L_e(\widetilde{W}) - M^{[\geq 0]}(\widetilde{W}) \\ &\quad - \mathbf{M}^{[\geq -1]}(W^{(3)}(\widetilde{W})) + \sum_{k=1}^{6} \lambda^{2(k-1)} \mathbf{a}_k \widetilde{\varphi}^k + \mathcal{R}_{\mathrm{mom}} \end{aligned} \tag{2.6.16}$$

$$\begin{aligned} 8\Delta \mathbf{\Phi}_1(\widetilde{\varphi}) &= 8\Delta\widetilde{\varphi} - 8\Delta_\gamma \widetilde{\varphi} + R(\gamma)\widetilde{\varphi} + \sum_{k=1}^{5} \lambda^{2k} \mathbf{b}_k \widetilde{\varphi}^k \\ &\quad - \left(|\sigma + L_\gamma W|_\gamma^2\right)^{(\geq 2)} - |\sigma + L_\gamma W|_\gamma^2 \left(\mathbf{c}_0 + \sum_{k\geq 1} \mathbf{c}_k \lambda^{2(k-1)} \widetilde{\varphi}^k \right) + \mathcal{R}_{\mathrm{Ham}}, \end{aligned} \tag{2.6.17}$$

and where $\mathcal{B}$ is defined in (2.5.39). Note that a fixed point of $\mathbf{\Phi}$ solves (2.6.12) and (2.6.13). In order to apply the Banach fixed point theorem and prove the existence of a fixed point, we prove in the next proposition that $\mathbf{\Phi}$ is well-defined and is a contraction.

Proposition 2.6.1. *If C_1 is large enough and ε is small enough, then $\mathbf{\Phi}$ is well-defined and is a contraction.*

Proof. Let $(\widetilde{\varphi}, \widetilde{W}) \in \mathcal{B} \times \mathcal{B}$. We start by estimating the $L^2_{\delta+2}$ norm of the RHS of (2.6.16):

$$\begin{aligned} \|\text{RHS of (2.6.16)}\|_{L^2_{\delta+2}} &\lesssim \left\| \mathrm{div}_e L_e(\widetilde{W}) - M^{[\geq 0]}(\widetilde{W}) \right\|_{L^2_{\delta+2}} + \left\| \mathbf{M}^{[\geq -1]}(W^{(3)}(\widetilde{W})) \right\|_{L^2_{\delta+2}} \\ &\quad + \left\| \sum_{k=1}^{6} \lambda^{2(k-1)} \mathbf{a}_k \widetilde{\varphi}^k \right\|_{L^2} + \|\mathcal{R}_{\mathrm{mom}}\|_{L^2} \\ &=: A + B + C + D \end{aligned}$$

where we omitted the weights for the last two terms since they are compactly supported. As in the proof of Proposition 2.5.1, we obtain $A \lesssim C_1 \varepsilon^2$. For B, we note that the operator $\mathbf{M}^{[\geq -1]}$ is linear and has bounded coefficients and involves up to two derivatives of $W^{(3)}(\widetilde{W})$, recall Lemma 2.4.7. Moreover, thanks to (2.6.3) $W^{(3)}(\widetilde{W})$ is compactly supported so we obtain $B \lesssim \left\| W^{(3)}(\widetilde{W}) \right\|_{H^2}$. Using (2.4.9) and (2.6.4) this implies $B \lesssim C_1 \varepsilon^2 + \varepsilon$.

For C, we simply estimate $\widetilde{\varphi}$ in L^∞ using the embedding $H^2_\delta \hookrightarrow L^\infty$ (see Proposition 2.1.1) and together with (2.6.7) this gives $C \lesssim C(C_1)\varepsilon^2$, where $C(C_1)$ denotes a numerical constant depending on C_1. Using (2.6.14), (2.5.17) and (2.6.7) again we also obtain $D \lesssim \varepsilon$. This discussion proves that

$$\|\text{RHS of (2.6.16)}\|_{L^2_{\delta+2}} \lesssim C(C_1)\varepsilon^2 + \varepsilon. \tag{2.6.18}$$

We now estimate the RHS of (2.6.17):

$$\begin{aligned} &\|\text{RHS of (2.6.17)}\|_{L^2_{\delta+2}} \\ &\lesssim \|\Delta\widetilde{\varphi} - \Delta_\gamma \widetilde{\varphi}\|_{L^2_{\delta+2}} + \|R(\gamma)\widetilde{\varphi}\|_{L^2_{\delta+2}} + \left\| \sum_{k=1}^{5} \lambda^{2k} \mathbf{b}_k \widetilde{\varphi}^k \right\|_{L^2} + \|\mathcal{R}_{\mathrm{Ham}}\|_{L^2_{\delta+2}} \\ &\quad + \left\| \left(|\sigma + L_\gamma W|_\gamma^2\right)^{(\geq 2)} \right\|_{L^2_{\delta+2}} + \left\| |\sigma + L_\gamma W|_\gamma^2 \left(\mathbf{c}_0 + \sum_{k\geq 1} \mathbf{c}_k \lambda^{2(k-1)} \widetilde{\varphi}^k \right) \right\|_{L^2_{\delta+2}} \\ &=: A + B + C + D + E + F \end{aligned}$$

where we omitted the weights for the third and sixth terms since they are compactly supported. For A we use the expansion defining γ, similarly as in (2.4.38):

$$A \lesssim \left\|(\gamma^{-1} - e^{-1})\partial^2\widetilde{\varphi}\right\|_{L^2_{\delta+2}} + \|\partial\gamma\partial\widetilde{\varphi}\|_{L^2_{\delta+2}} \lesssim C_1\varepsilon^2$$

where we bound the metric coefficients and their derivatives in L^∞ using (2.5.40) and (2.5.41). The terms B and C only contains $\widetilde{\varphi}$ with zero derivatives, which we simply bound in L^∞ using $H^2_\delta \hookrightarrow L^\infty$. We then use Lemma 2.4.2 and (2.6.9) to obtain $B + C \lesssim C(C_1)\varepsilon^2$. Similar arguments lead to $D \lesssim \varepsilon$.

We now estimate E and F. It involves the TT-tensor σ but thanks to the estimate (2.5.45) we can put it in L^∞ and thus focus on $L_\gamma W$. For the same reason, we neglect $W^{(2)}$ and $W^{(3)}(\widetilde{W})$. Since the term

$$\mathbf{c}_0 + \sum_{k\geq 1} \mathbf{c}_k \lambda^{2(k-1)}\widetilde{\varphi}^k$$

can be bounded in L^∞ by $C(C_1)\varepsilon$ (using (2.6.10)-(2.6.11) and $H^2_\delta \hookrightarrow L^\infty$ for the powers of $\widetilde{\varphi}$), in order to estimate E and F it is enough to estimate $\left\|(L_\gamma\widetilde{W})^2\right\|_{L^2_{\delta+2}}$. Since $L_\gamma\widetilde{W}$ contains derivatives of γ we can't directly use the product law $H^1_{\delta+1} \times H^1_{\delta+1} \hookrightarrow L^2_{\delta+2}$ of Proposition 2.1.1 without losing one λ power. Instead we expand

$$\left\|(L_\gamma\widetilde{W})^2\right\|_{L^2_{\delta+2}} \lesssim \left\|(\partial\widetilde{W})^2\right\|_{L^2_{\delta+2}} + \left\|(\partial\gamma)^2(\widetilde{W})^2\right\|_{L^2_{\delta+2}} + \left\|\partial\gamma\widetilde{W}\partial\widetilde{W}\right\|_{L^2_{\delta+2}}.$$

For the first term we use the product law $H^1_{\delta+1} \times H^1_{\delta+1} \hookrightarrow L^2_{\delta+2}$ of Proposition 2.1.1. For the second and third terms, we bound $\partial\gamma$ in L^∞ (recall (2.5.41)) and use the product laws $H^2_\delta \times H^2_\delta \hookrightarrow L^2_{\delta+2}$ and $H^2_\delta \times H^1_{\delta+1} \hookrightarrow L^2_{\delta+2}$. We obtain $\left\|(L_\gamma\widetilde{W})^2\right\|_{L^2_{\delta+2}} \lesssim C(C_1)\varepsilon^2$ and

$$E + F \lesssim C(C_1)\varepsilon^2 + \varepsilon.$$

This discussion proves that

$$\|\text{RHS of (2.6.17)}\|_{L^2_{\delta+2}} \lesssim C(C_1)\varepsilon^2 + \varepsilon. \tag{2.6.19}$$

Using the first part of Proposition 2.1.2, (2.6.18) and (2.6.19) prove that there exists a unique $\left(\mathbf{\Phi}_1(\widetilde{\varphi}), \mathbf{\Phi}_2(\widetilde{W})\right) \in H^2_\delta \times H^2_\delta$ solving (2.6.16)-(2.6.17) and satisfying

$$\|\mathbf{\Phi}_1(\widetilde{\varphi})\|_{H^2_\delta} + \left\|\mathbf{\Phi}_2(\widetilde{W})\right\|_{H^2_\delta} \lesssim C(C_1)\varepsilon^2 + \varepsilon.$$

Therefore, if we take C_1 larger than the numerical constant appearing in these estimates and ε small compared to C_1, then $\left(\mathbf{\Phi}_1(\widetilde{\varphi}), \mathbf{\Phi}_2(\widetilde{W})\right) \in \mathcal{B} \times \mathcal{B}$. This shows that $\mathbf{\Phi}$ is well-defined.

In order to show that $\mathbf{\Phi}$ is a contraction we consider the equations satisfied by the differences $\mathbf{\Phi}_1(\widetilde{\varphi}_a) - \mathbf{\Phi}_1(\widetilde{\varphi}_b)$ and $\mathbf{\Phi}_2(\widetilde{W}_a) - \mathbf{\Phi}_2(\widetilde{W}_b)$, where $(\widetilde{\varphi}_a, \widetilde{W}_a)$ and $(\widetilde{\varphi}_b, \widetilde{W}_b)$ are two elements of $\mathcal{B} \times \mathcal{B}$. Together with non-linear inequalities of the form

$$\left|x^k - y^k\right| \lesssim \sup_{0\leq p,q\leq k-1} \{|x|^p, |y|^q\} \times |x - y|$$

we can mimick the previous arguments leading to (2.6.18) and (2.6.19) and prove that by taking C_1 larger and ε smaller if necessary the map $\mathbf{\Phi}$ is a contraction. We omit the details. □

The Banach fixed point theorem then implies that there exists $(\widetilde{\varphi}, \widetilde{W}) \in \mathcal{B} \times \mathcal{B}$ solving (2.6.12) and (2.6.13). We can also prove that $\widetilde{\varphi}$ and $\widetilde{W}$ enjoy higher regularity, as we did for Y in Section 2.5.4.2. We obtain $\widetilde{\varphi}, \widetilde{W} \in H^{N-3}_\delta$ with

$$\|\widetilde{\varphi}\|_{H^{k+2}_\delta} + \left\|\widetilde{W}\right\|_{H^{k+2}_\delta} \lesssim \frac{\varepsilon}{\lambda^k} \tag{2.6.20}$$

for $k \in [\![0, N-5]\!]$. This concludes the construction of high-frequency solutions to (2.3.3)-(2.3.4).

2.7 Proof of the main theorem

In this section we conclude the proof of Theorem 2.2.1. The solution of the constraint equations $(\bar{g}_\lambda, K_\lambda)$ we constructed through the conformal method is given by

$$\bar{g}_\lambda = \varphi^4 \gamma, \tag{2.7.1}$$

$$K_\lambda = \varphi^{-2}(\sigma + L_\gamma W) + \frac{1}{3}\varphi^4 \gamma \tau \tag{2.7.2}$$

where γ, τ, σ, W and φ are the parameters and unknowns of the conformal method and are defined in the previous sections. Let us check that the two previous expressions match the expressions of Theorem 2.2.1 and the estimates therein.

2.7.1 The metric $\bar{g}_\lambda$ and proof of (2.2.13)

We start with the induced metric. Thanks to (2.3.9) and (2.7.1) we first have

$$\bar{g}_\lambda = \bar{g}_0 + \lambda \gamma^{(1)} + O\left(\lambda^2\right)$$

which matches (2.2.8) using (2.4.2) and (2.4.8). If we now look at the order λ^2 or higher in $\varphi^4\gamma$, we see that it is composed of oscillating terms and terms satisfying better estimates:

$$\left(\varphi^4\gamma\right)^{(\geq 2)} = 4\varphi^{(2)}\bar{g}_0 + 4\widetilde{\varphi}\bar{g}_0 + \gamma^{(2)} + \lambda\left(4\varphi^{(3)}\bar{g}_0 + 4\left(\widetilde{\varphi} + \varphi^{(2)}\right)\gamma^{(1)}\right) + O\left(\lambda^2\right) \tag{2.7.3}$$

where the $O\left(\lambda^2\right)$ is a polynomial in terms of $\varphi^{(2)}$, $\widetilde{\varphi}$, $\varphi^{(3)}$, $\bar{g}_0$, $\gamma^{(1)}$ and $\gamma^{(2)}$. Using (2.5.6), (2.4.3) and (2.5.7)-(2.5.8) we see that

$$\varphi^{(2)}\bar{g}_0 + \gamma^{(2)} = \sin\left(\frac{u_0}{\lambda}\right)\bar{F}^{(2,1)} + \cos\left(\frac{2u_0}{\lambda}\right)\bar{F}^{(2,2)}.$$

Therefore, by setting

$$\bar{\mathfrak{h}}_\lambda = \left(\varphi^4\gamma\right)^{(\geq 2)} - 4\varphi^{(2)}\bar{g}_0 - 4\gamma^{(2)}$$

we prove that $\bar{g}_\lambda$ is indeed given by the expression (2.2.8). We now prove estimate (2.2.13). Thanks to (2.7.3) we have

$$\bar{\mathfrak{h}}_\lambda = 4\widetilde{\varphi}\bar{g}_0 + \lambda\left(4\varphi^{(3)}\bar{g}_0 + 4\left(\widetilde{\varphi} + \varphi^{(2)}\right)\gamma^{(1)}\right) + O\left(\lambda^2\right). \tag{2.7.4}$$

The regularity of each term in $\bar{\mathfrak{h}}_\lambda$ (recall (2.6.20)) and the decay of $\widetilde{\varphi}$ and $\bar{g}_0$ at infinity imply easily that the amount of derivatives together with the weights in (2.2.13) are allowed. The only part of (2.2.13) that remains to be checked is the λ behaviour. From this perspective, $\varphi^{(3)}$, $\varphi^{(2)}$ and $\gamma^{(1)}$ are the worse terms since they lose one λ power for each derivative. As they are already multiplied by λ in (2.7.4), this concludes the justification of (2.2.13).

2.7.2 The tensor K_λ and proof of (2.2.14)

For the tensor K_λ, we first prove that (2.7.2) matches the expression (2.2.9). Since $\varphi = 1 + O\left(\lambda^2\right)$, we have $\varphi^{-2} = 1 + O\left(\lambda^2\right)$ and $\varphi^4 = 1 + O\left(\lambda^2\right)$. Therefore from (2.7.2) we obtain

$$K_\lambda = \sigma^{(0)} + (L_\gamma W)^{(0)} + \lambda\left(\sigma^{(1)} + (L_\gamma W)^{(1)} + \frac{1}{3}\bar{g}_0\tau^{(1)}\right) + O\left(\lambda^2\right)$$

We now use the ansatz for W (see (2.3.10)) and the expansion of Lemma 2.4.5 to obtain $(L_\gamma W)^{(0)} = 0$ and $(L_\gamma W)^{(1)} = \mathbf{K}^{[-1]}(W^{(2)})$. This gives

$$\begin{aligned} K_\lambda &= \sigma^{(0)} + \lambda\left(\sigma^{(1)} + \mathbf{K}^{[-1]}(W^{(2)}) + \frac{1}{3}\bar{g}_0\tau^{(1)}\right) + O\left(\lambda^2\right) \\ &= K_\lambda^{(1)} + \lambda K_\lambda^{(1)} + O\left(\lambda^2\right) \end{aligned}$$

where we used the definition of $\sigma^{(0)}$ and $\sigma^{(1)}$, see (2.5.20) and (2.5.21). Therefore, the solution K_λ matches the expression (2.2.9). The remainder $K_\lambda^{(\geq 2)}$ satisfies

$$\begin{aligned} K_\lambda^{(\geq 2)} = {}& \sigma^{(2)}(Y) + L_\gamma Y + \mathbf{K}^{[\geq 0]}(W^{(2)}) + \mathbf{K}^{[\geq -1]}(W^{(3)}(\widetilde{W})) + L_\gamma \widetilde{W} \\ & - 2\left(\widetilde{\varphi} + \varphi^{(2)}\right)\sigma^{(0)} + \frac{1}{3}\gamma^{(1)}\tau^{(1)} + O\left(\lambda^3\right). \end{aligned} \tag{2.7.5}$$

The estimate (2.2.14) then follows from estimating directly all the oscillating terms in (2.7.5) and using (2.5.44) and (2.6.20) for $L_\gamma Y$, $L_\gamma \widetilde{W}$ or $\widetilde{\varphi}$. This concludes the proof of Theorem 2.2.1 and this chapter.

Chapter 3

Local existence in generalised wave gauge

3.1 Introduction

In this chapter, we are interesting in special solutions of the Einstein vacuum equations

$$R_{\mu\nu}(g) = 0 \tag{3.1.1}$$

where $R_{\mu\nu}(g)$ is the Ricci tensor of the Lorentzian metric g defined on a manifold $\mathcal{M}$. More precisely, we construct a sequence of metrics $(g_\lambda)_\lambda$ solutions of (3.1.1) converging when $\lambda \longrightarrow 0$ in a particular sense to a background metric g_0 solution of the Einstein equations coupled to a null dust (equations (3.1.2)-(3.1.3)-(3.1.4) below). As is standard when solving the Einstein equations, we need to choose a gauge. In this chapter, we work in generalised wave gauge, i.e we rewrite (3.1.1) as a system of quasi-linear wave equations for the metric coefficients. The intital data for this system are given by the results in Chapter 2 and in particular they solve the constraint equations on $\mathbb{R}^3$.

The main feature of the sequence $(g_\lambda)_\lambda$ constructed is its high-frequency character. Concretely, the metric g_λ is defined by a high-frequency expansion (λ being a small wavelength) with a non-oscillating remainder in the spirit of geometric optics and the work of Choquet-Bruhat in [CB69]. The system (3.1.1) is then recast as a hierarchy of equations for each term of the ansatz: transport equations along the background rays for the oscillating terms and a wave equation for the remainder. Our result constitutes the first rigorous justification of the geometric optics approximation for the Einstein vacuum equations in $3+1$ (see [HL18b] for a similar construction in $2+1$) and we highlight now its main challenges.

The quasi-linear aspect of (3.1.1) implies a loss of derivatives in the coupling between the wave equation for the remainder and one of the transport equation. Regaining this derivatives is the main analytical challenge of our proof. It heavily relies on the expansion defining g_λ and on the use of the background null structure, i.e the foliation associated to a solution of the eikonal equation for the background metric g_0.

The quadratic terms in (3.1.1) cause the creation of harmonics, which we need to describe precisely in order to define our high-frequency ansatz. We rely on the weak polarized null structure of these terms. This allows us to recover a linear behaviour for the propagation of the waves along the rays, despite the non-linear nature of (3.1.1). This is known as *transparency* in the geometric optics literature, see [Mé09].

Due to their gauge invariance, the Einstein vacuum are not hyperbolic. In the case of high-frequency waves considered here, this implies the presence of *polarization* conditions.

They play a role in the control of the semi-linear and quasi-linear terms presented above but they need to be propagated by the evolution. As opposed to [CB69], we propagate the polarization conditions of the second order perturbation in the high-frequency ansatz relying on the contracted Bianchi identities rather than on the transport equations along the rays.

The remainder of this introduction is structured as follows:

- we introduce our notations and the usual material in Section 3.1.1,
- we gather in Section 3.1.2 all the analytical and formal properties of the background spacetime,
- in Section 3.1.3 we define the class of high-frequency initial data we consider,
- finally in Section 3.1.4 we state the main result of this chapter, Theorem 3.1.2.

3.1.1 Geometry, function spaces and high-frequency notations

We will work on the manifold $\mathcal{M} := [0,1] \times \mathbb{R}^3$. For $t \in [0,1]$, we denote by Σ_t the spacelike hypersurface $\{t\} \times \mathbb{R}^3$. The greek indices go from 0 to 3 and will always refer to the usual global coordinates system (t, x^1, x^2, x^3) on $\mathcal{M}$, while latin indices go from 1 to 3 and correspond to the three spatial directions.

Let T and S be two symmetric 2-tensors on $\mathcal{M}$ and let g be a Lorentzian metric on $\mathcal{M}$. We define a scalar product by

$$|T \cdot S|_g = g^{\alpha\beta} g^{\mu\nu} T_{\alpha\mu} S_{\beta\nu}$$

with associated norm $|T|_g^2 = |T \cdot T|_g$. The trace of T with respect to g is defined by $\mathrm{tr}_g T = g^{\alpha\beta} T_{\alpha\beta}$.

If f is a scalar function on $\mathcal{M}$, we denote by ∇f a spatial derivative of f and by ∂f a time derivative $\partial_t f$ or ∇f. The wave operator associated to a Lorentzian metric is defined by

$$\Box_g f = g^{\alpha\beta} \left(\partial_\alpha \partial_\beta f - \Gamma^\mu_{\alpha\beta} \partial_\mu f \right)$$

where $\Gamma^\mu_{\alpha\beta}$ are the Christoffel symbols. The principal part of this operator is denoted $\tilde{\Box}_g$, i.e $\tilde{\Box}_g f = g^{\alpha\beta} \partial_\alpha \partial_\beta f$.

On each slice Σ_t we consider the usual functions spaces L^p and $W^{k,p}$, by which we always mean $L^p(\Sigma_t)$ and $W^{k,p}(\Sigma_t)$ for some t depending on the context. We will also use the weighted Sobolev spaces defined in Section 2.1.2.2 of Chapter 2, where we also state their product law and the usual embeddings.

There won't be any issue of regularity in the time variable so if f is a scalar function defined on $\mathcal{M}$, then $\|f\|_X \leq C$ without any precision on t will always mean

$$\sup_{t \in [0,1]} \|f\|_{X(\Sigma_t)} \leq C$$

where X is one of the function spaces defined here.

As in Chapter 2, we deal here with high-frequency quantities and we follow the same conventions as they are stated in Section 2.1.2.4.

3.1.2 The background spacetime

The background quantities are defined on $\mathcal{M}$ and are composed of a metric g_0, an optical function u_0 and a density F_0. They are solution of the Einstein null dust system on $\mathcal{M}$:

$$R_{\mu\nu}(g_0) = F_0^2 \partial_\mu u_0 \partial_\nu u_0, \tag{3.1.2}$$

$$g_0^{-1}(\mathrm{d}u_0, \mathrm{d}u_0) = 0, \tag{3.1.3}$$

$$-2L_0 F_0 + (\Box_{g_0} u_0) F_0 = 0, \tag{3.1.4}$$

where $R_{\mu\nu}(g_0)$ is the Ricci tensor associated to g_0 and L_0 is the spacetime gradient of u_0, i.e

$$L_0 = -g_0^{\alpha\beta} \partial_\alpha u_0 \partial_\beta. \tag{3.1.5}$$

By (3.1.3), L_0 is null and geodesic, i.e

$$g_0(L_0, L_0) = 0 \quad \text{and} \quad \mathbf{D}_{L_0} L_0 = 0,$$

where $\mathbf{D}$ denotes throughout this chapter the covariant derivative associated to g_0. We also assume that g_0 satisfy in the standard coordinates (t, x^1, x^2, x^3) the wave condition

$$g_0^{\mu\nu} \Gamma(g_0)_{\mu\nu}^\alpha = 0. \tag{3.1.6}$$

Note that this condition implies that the wave operator associated to g_0 reduces to its principal part, i.e $\Box_{g_0} = \tilde{\Box}_{g_0}$. A standard computation shows that under the wave condition (3.1.6) the Einstein type equation (3.1.2) rewrites as

$$\tilde{\Box}_{g_0}(g_0)_{\alpha\beta} = P_{\alpha\beta}(g_0)(\partial g_0, \partial g_0) - 2F_0^2 \partial_\alpha u_0 \partial_\beta u_0. \tag{3.1.7}$$

where $P_{\alpha\beta}(g_0)(\partial g_0, \partial g_0)$ is a quadratic non-linearity (see (3.2.2) for its exact expression). We make the following assumptions on the background quantities (g_0, u_0, F_0). In what follows, $N \geq 10$, $\delta > -\frac{3}{2}$ and $\varepsilon > 0$ is our smallness threshold.

- **Assumptions on the metric** g_0. There exists a constant $\varepsilon > 0$ such that for all $t \in [0, 1]$

$$\|g_0 - m\|_{H_\delta^{N+1}} + \|\partial_t g_0\|_{H_{\delta+1}^N} + \left\|\partial_t^2 g_0\right\|_{H_{\delta+2}^{N-1}} \leq \varepsilon \tag{3.1.8}$$

  where m is the Minkowski metric on $\mathcal{M}$. For convenience, we also assume that ∂_t is the unit normal to Σ_0 of g_0, and that the second fundamental form of Σ_0 is traceless. The former assumption simplifies the construction of the initial data for the high-frequency metric (see Section 3.4.6) and the latter is only use in the proof of Theorem 2.2.1 (see Chapter 2) in order to solve the constraint equations. The estimates (3.1.8) means that we work in the asymptotically flat setting.

- **Assumptions on the optical function** u_0. We assume that there exists a constant non-zero vector field $\mathfrak{z} = (\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3)$ such that

$$\|\nabla u_0 - \mathfrak{z}\|_{H_{\delta+1}^N} \leq \varepsilon \tag{3.1.9}$$

  where $\nabla u_0 = (\partial_1 u_0, \partial_2 u_0, \partial_3 u_0)$ is the spatial Euclidean gradient of u_0. This implies that the level sets of u_0 restricted to any Σ_t are asymptotically planes in $\mathbb{R}^3$ and that if ε is small enough there exists $c > 0$ such that

$$\inf_{x \in \mathbb{R}^3} |\nabla u_0|(x) > c. \tag{3.1.10}$$

This in turn implies that u_0 has no critical point. Finally, we assume that L_0 defined by (3.1.5) is futur-directed and since ∂_t is assumed to be the unit normal of g_0 to Σ_0 and u_0 solves (3.1.3), this implies

$$\partial_t u_0 = |\nabla u_0|_{\bar{g}_0} \tag{3.1.11}$$

on Σ_0.

- **Assumptions on the density** F_0. We assume that F_0 is initially compactly supported, i.e there exists $R > 0$ such that $F_0 \restriction \Sigma_0$ is supported in $\{|x| \leq R\}$. Since L_0 is null and geodesic, (3.1.4) shows that

$$\mathrm{supp}\,(F_0) \subset J_0^+ \left(\{|x| \leq R\}\right)$$

where J_0^+ denotes the causal future associated to g_0. We can assume for simplicity that

$$J_0^+ \left(\{|x| \leq R\}\right) \subset \{(t,x) \in \mathcal{M} \mid |x| \leq C_{\mathrm{supp}} R\}$$

for some constant $C_{\mathrm{supp}} > 0$. Finally we assume the following estimate:

$$\|F_0\|_{H^N} \leq \varepsilon. \tag{3.1.12}$$

In this chapter we don't prove the existence of the background solution (g_0, F_0, u_0) on $\mathcal{M}$ since it follows from standard arguments. We will denote by C_0 any numerical constant depending on the background estimates stated here, that is depending on δ,N, R or $\mathfrak{z}$.

In addition to the usual coordinates on $[0,1] \times \mathbb{R}^3$ we will use a frame linked to the optical function u_0, i.e the null frame, which is defined as follows. We denote by $\mathcal{H}_u$ the level sets of u_0, i.e

$$\mathcal{H}_u = \{(t,x) \in \mathcal{M} \mid u_0(t,x) = u\}.$$

Since u_0 is a solution of the eikonal equation (3.1.3), each $\mathcal{H}_u$ is a null hypersurface generated by the geodesic vector field L_0. Thanks to (3.1.10) they induce a foliation of the spacetime. For $t \in [0,1]$ and u in the image of u_0 we define the following 2-surfaces

$$P_{t,u} = \Sigma_t \cap \mathcal{H}_u.$$

Thanks to (3.1.9), each $P_{t,u}$ have the topology of a plane in $\mathbb{R}^3$. We denote by $\bar{g}_0$ the induced metric on Σ_t and $\mathring{g}_0$ the induced metric on $P_{t,u}$. We consider a null vector field $\underline{L}_0$ such that $g_0(\underline{L}_0, L_0) = -2$ and an orthonormal frame (e_1, e_2) of $TP_{t,u}$ for $\mathring{g}_0$. This defines the *background null frame* $(L_0, \underline{L}_0, e_1, e_2)$, and such a choice is always possible, at least locally. We use the latin upper case letters as indices for the frame (e_1, e_2) on $TP_{t,u}$. We consider the subset

$$\mathcal{T}_0 = \{L_0, e_1, e_2\} \tag{3.1.13}$$

of the background null frame. The use of such a frame originates in the seminal work of Christodoulou and Klainerman on the stability of Minkowski spacetime (see [CK93]) and a concise presentation can be found in [Sze18].

Further geometric objects related to the background null frame will be needed for the proof of Theorem 3.1.2, linked to the commutator $[L_0, \Box_{g_0}]$. We refer to Lemma 3.2.1 for the main result on this commutator and to Appendix 3.A for its proof.

3.1.3 Initial data

We reproduce here the main result of Chapter 2 where we construct high-frequency solutions $(\bar{g}_\lambda, K_\lambda)$ of the constraint equations on $\mathbb{R}^3$, that is

$$R(\bar{g}_\lambda) + (\mathrm{tr}_{\bar{g}_\lambda} K_\lambda)^2 - |K_\lambda|^2_{\bar{g}_\lambda} = 0, \tag{3.1.14}$$

$$-\mathrm{div}_{\bar{g}_\lambda} K_\lambda + \mathrm{dtr}_{\bar{g}_\lambda} K_\lambda = 0. \tag{3.1.15}$$

Theorem 3.1.1. *Let (g_0, u_0, F_0) be the solution of the Einstein-null dust system described in Section 3.1.2, and let $\varepsilon > 0$ be the smallness threshold. There exists $\varepsilon_0 = \varepsilon_0(\delta, R) > 0$ such that if $0 < \varepsilon \leq \varepsilon_0$, there exists for all $\lambda \in (0,1]$ a solution $(\bar{g}_\lambda, K_\lambda)$ solution of the constraint equations (3.1.14)-(3.1.15) on $\mathbb{R}^3$ of the form*

$$\bar{g}_\lambda = \bar{g}_0 + \lambda \cos\left(\frac{u_0}{\lambda}\right) \bar{F}^{(1)} + \lambda^2 \left(\sin\left(\frac{u_0}{\lambda}\right) \bar{F}^{(2,1)} + \cos\left(\frac{2u_0}{\lambda}\right) \bar{F}^{(2,2)} \right) + \lambda^2 \bar{\mathfrak{h}}_\lambda, \tag{3.1.16}$$

$$K_\lambda = K_\lambda^{(0)} + \lambda K_\lambda^{(1)} + \lambda^2 K_\lambda^{(\geq 2)}, \tag{3.1.17}$$

with

$$K_\lambda^{(0)} = K_0 + \frac{1}{2} \sin\left(\frac{u_0}{\lambda}\right) |\nabla u_0|_{\bar{g}_0} \bar{F}^{(1)}, \tag{3.1.18}$$

$$\begin{aligned} \left(K_\lambda^{(1)}\right)_{ij} = & -\frac{1}{2} \cos\left(\frac{u_0}{\lambda}\right) \left(-N_0 \bar{F}^{(1)}_{ij} + (\partial_t + N_0)^\rho \Gamma(g_0)^k_{\rho(i} \bar{F}^{(1)}_{j)k} + \frac{1}{2|\nabla u_0|_{\bar{g}_0}} (\Box_{g_0} u_0) \bar{F}^{(1)}_{ij} \right) \\ & - \frac{1}{2} |\nabla u_0|_{\bar{g}_0} \left(\cos\left(\frac{u_0}{\lambda}\right) \bar{F}^{(2,1)}_{ij} - 2 \sin\left(\frac{2u_0}{\lambda}\right) \bar{F}^{(2,2)}_{ij} \right), \end{aligned} \tag{3.1.19}$$

where K_0 is the second fundamental form of Σ_0 for g_0. Moreover:

(i) the tensors $\bar{F}^{(1)}$, $\bar{F}^{(2,1)}$ and $\bar{F}^{(2,2)}$ are supported in $\{|x| \leq R\}$ and there exists $C_{\mathrm{cons}} = C_{\mathrm{cons}}(\delta, R) > 0$ such that

$$\left\| \bar{F}^{(1)} \right\|_{H^N} + \left\| \bar{F}^{(2,1)} \right\|_{H^{N-1}} + \left\| \bar{F}^{(2,2)} \right\|_{H^{N-1}} \leq C_{\mathrm{cons}} \varepsilon, \tag{3.1.20}$$

(ii) the tensor $\bar{F}^{(1)}$ is $\bar{g}_0$-traceless, tangential to $P_{0,u}$ and satisfies

$$\left| \bar{F}^{(1)} \right|^2_{\bar{g}_0} = 8 F_0^2, \tag{3.1.21}$$

(iii) the tensors $\bar{\mathfrak{h}}_\lambda$ and $K_\lambda^{(\geq 2)}$ belong to the spaces H^5_δ and $H^4_{\delta+1}$ respectively and satisfy

$$\max_{r \in [\![0,4]\!]} \lambda^r \left\| \nabla^{r+1} \bar{\mathfrak{h}}_\lambda \right\|_{L^2_{\delta+r+1}} \leq C_{\mathrm{cons}} \varepsilon, \tag{3.1.22}$$

$$\max_{r \in [\![0,4]\!]} \lambda^r \left\| \nabla^r K_\lambda^{(\geq 2)} \right\|_{L^2_{\delta+r+1}} \leq C_{\mathrm{cons}} \varepsilon. \tag{3.1.23}$$

3.1.4 Local existence

The following theorem is the main result of this chapter.

Theorem 3.1.2. *Let (g_0, u_0, F_0) be the solution of the Einstein-null dust system described in Section 3.1.2, and let $\varepsilon > 0$ be the smallness threshold. There exists $\lambda_0 \in \mathbb{R}$ and $\varepsilon_0 = \varepsilon_0(\delta, R) > 0$ such that if $0 < \varepsilon \leq \varepsilon_0$, there exists for all $\lambda \in (0, \lambda_0]$ a solution g_λ of the form*

$$g_\lambda = g_0 + \lambda g^{(1)}\left(\frac{u_0}{\lambda}\right) + \lambda^2 g^{(2)}\left(\frac{u_0}{\lambda}\right) + \lambda^2 \tilde{\mathfrak{h}}_\lambda \tag{3.1.24}$$

to (3.1.1) on $[0,1] \times \mathbb{R}^3$ in generalised wave gauge. Moreover:

(i) the tensors $g^{(i)}$ for $i = 1, 2$ are supported in $J_0^+(\{|x| \leq R\})$ and are periodic and smooth functions of the argument $\frac{u_0}{\lambda}$,

(ii) there exists $C = C(\delta, R) > 0$ such that the following estimates hold

$$\max_{r\in[\![0,10]\!]} \lambda^r \left\| \nabla^r g^{(1)} \right\|_{L^2} + \max_{r\in[\![0,6]\!]} \lambda^r \left\| \nabla^r g^{(2)} \right\|_{L^2} + \max_{r\in[\![1,5]\!]} \lambda^r \left\| \nabla^{r+1} \tilde{\mathfrak{h}}_\lambda \right\|_{L^2_{\delta+r}} \leq C\varepsilon, \tag{3.1.25}$$

(iii) the tensor $g^{(1)}_{\mu\nu} = \cos\left(\frac{u_0}{\lambda}\right) F^{(1)}_{\mu\nu}$ satisfy the following polarization and energy conditions

$$g_0^{\mu\nu}\left(\partial_\mu u_0 F^{(1)}_{\sigma\nu} - \frac{1}{2}\partial_\sigma u_0 F^{(1)}_{\mu\nu}\right) = 0, \tag{3.1.26}$$

$$\frac{1}{8}\left|F^{(1)}\right|^2_{g_0} - \frac{1}{16}\left(\mathrm{tr}_{g_0} F^{(1)}\right)^2 = F_0^2. \tag{3.1.27}$$

Before we present the strategy behind the proof of Theorem 3.1.2 in Section 3.2, let us make some remarks on this result.

- The notation $g^{(i)}\left(\frac{u_0}{\lambda}\right)$ is used in (3.1.24) to emphasize the oscillating character of $g^{(i)}$, but these tensors also depends on the spacetime variables and this notation actually stands for the following functions on the spacetime

$$g^{(i)}_{\alpha\beta}\left(\frac{u_0}{\lambda}\right)(t,x) = g^{(i)}_{\alpha\beta}\left(t, x, \frac{u_0(t,x)}{\lambda}\right)$$

  where $(t,x) \in [0,1] \times \mathbb{R}^3$.

- The exact expression of the ansatz for g_λ is given in Section 3.3.1. Compared to (3.1.24), we will then describe precisely the oscillating behaviour of each $g^{(i)}$ for $i = 1, 2$, i.e give a finite frequency decomposition. Note that in Theorem 3.1.2 we already gave the frequency decomposition of $g^{(1)}$, that is $g^{(1)} = \cos\left(\frac{u_0}{\lambda}\right) F^{(1)}$.

- As explained in depth in the introduction, our main goal is to consider high-frequency limits, that is $\lambda \longrightarrow 0$. Since the smallness threshold ε_0 is independent from λ, our construction is valid in the high-frequency limit and the estimates

$$\|g_\lambda - g_0\|_{L^\infty} \lesssim \lambda,$$
$$\|\partial(g_\lambda - g_0)\|_{L^2} \lesssim 1$$

  hold even for small λ, as well as the time of existence equal to 1. From this perspective, the restriction $\lambda \in (0, \lambda_0]$ with λ_0 potentially very small is meaningless and only plays a role in the improvement of one of our bootstrap assumptions (see Proposition 3.6.1).

- The tensor $g^{(2)}$ also satisfies polarization conditions similar to (3.1.26) but with complicated RHS depending on $g^{(1)}$. See Section 3.4.4.2 for more details.

- The generalised wave condition defining our gauge has a complicated expression and we choose not to present it in Theorem 3.1.2. In particular, it depends on the exact ansatz for g_λ and the coupling between the transport equations for the waves and the wave equation for the remainder. See Section 3.4.4.1 for its exact expression.

- The oscillating terms in (3.1.24) depend on the phase $\frac{u_0}{\lambda}$, where u_0 is the solution of the *background* eikonal equation, i.e (3.1.3). We emphasize the fact that we don't solve the eikonal equation for the metric g_λ, as in [HL18a] but as opposed to [LR20]. This would require the construction of a null foliation in 3+1 dimensions and hence would complicate our proof.

3.2 Strategy of proof

In this section, we sketch the proof of Theorem 3.1.2. The challenge we face is twofold: first we need to define the high-frequency ansatz for g_λ in a fully non-linear context, and then we need to solve the hierarchy of equations obtained which in particular amounts to solve an apparent loss of derivatives.

3.2.1 The high-frequency ansatz

In any coordinates system, the Ricci tensor of a Lorentzian metric g reads

$$2R_{\alpha\beta}(g) = -\tilde{\Box}_g g_{\alpha\beta} + H^\rho \partial_\rho g_{\alpha\beta} + g_{\rho(\alpha}\partial_{\beta)} H^\rho + P_{\alpha\beta}(g)(\partial g, \partial g). \tag{3.2.1}$$

Let us detail the different terms in this expression.

- $\tilde{\Box}_g$ stands for the principal part of the wave operator, i.e $g^{\mu\nu}\partial_\mu\partial_\nu$. Because of this term, the Einstein vacuum equations (3.1.1) are quasi-linear. Moreover, if we think of (3.1.1) as a system of second order partial differential operators for the metric coefficients, the $-\tilde{\Box}_g g_{\alpha\beta}$ term represents the diagonal part of the Ricci operator.

- The terms involving $H^\rho = g^{\mu\nu}\Gamma^\rho_{\mu\nu}$ are the gauge part of the Ricci operator and they involve second order derivatives of the metric coefficients. They represent the non-diagonal part of the Ricci operator and working in generalised wave gauge precisely means that we prescribe the value of H^ρ in order to solve the loss of hyperbolicity it causes.

- The quadratic non-linearity $P_{\alpha\beta}(g)(\partial g, \partial g)$ is given in coordinates by

$$\begin{aligned} P_{\alpha\beta}(g)(\partial g, \partial g) &= g^{\mu\rho}g^{\nu\sigma}\partial_{(\alpha} g_{\rho\sigma}\partial_\mu g_{\beta)\nu} - \frac{1}{2} g^{\mu\rho}g^{\nu\sigma}\partial_\alpha g_{\rho\sigma}\partial_\beta g_{\mu\nu} \\ &\quad - g^{\mu\rho}g^{\nu\sigma}\partial_\rho g_{\alpha\nu}\partial_\sigma g_{\beta\mu} + g^{\mu\rho}g^{\nu\sigma}\partial_\rho g_{\sigma\alpha}\partial_\mu g_{\nu\beta}. \end{aligned} \tag{3.2.2}$$

  It enjoys what we call the weak polarized null condition defined in Section 3.2.1.3.

While being highly non-linear, the Ricci tensor and the three parts defining it all enjoy some special structure at the heart of our construction. The precise ansatz we choose is

$$g_\lambda = g_0 + \lambda g^{(1)} + \lambda^2 \left(g^{(2)} + \mathfrak{h}_\lambda \right) + \lambda^3 g^{(3)} \tag{3.2.3}$$

The remainder $\mathfrak{h}_\lambda$ is a symmetric 2-tensor and each $g^{(i)}_{\alpha\beta}$ is a finite linear combination of terms of the form $\mathrm{T}\left(\frac{u_0}{\lambda}\right) S_{\alpha\beta}$ for S a symmetric 2-tensor and T a trigonometric function, that is

$$\mathrm{T} \in \{ s \longmapsto \cos(ks) \mid k \in \mathbb{N} \} \cup \{ s \longmapsto \sin(ks) \mid k \in \mathbb{N} \}. \tag{3.2.4}$$

Without loss of generality, we can assume that $g^{(1)}$ has the form $g^{(1)}_{\alpha\beta} = \cos\left(\frac{u_0}{\lambda}\right) F^{(1)}_{\alpha\beta}$, as stated in Theorem 3.1.2. The Ricci tensor being non-linear, we must incorporate the creation of harmonics into g_λ. Even though terms like $P_{\alpha\beta}(g)(\partial g, \partial g)$ are of the form $g^{-1}g^{-1}\partial g \partial g$ and thus corresponds to an interaction of four oscillating terms, the derivatives imply that this term corresponds actually to an interaction of only two oscillating terms, at least at leading order. Therefore, we expect $g^{(2)}$ to oscillate with frequency θ and 2θ. More precisely, by analogy with our toy model in Chapter 4, $g^{(2)}$ oscillates like $\sin(\theta)$ and $\cos(2\theta)$.

The Ricci tensor involves second derivatives of the metric, and $R_{\mu\nu}(g_\lambda)$ admits *a priori* a formal expansion of the form

$$R_{\mu\nu}(g_\lambda) = \frac{1}{\lambda} R^{(-1)}_{\mu\nu} + R^{(0)}_{\mu\nu} + \lambda R^{(1)}_{\mu\nu} + \cdots .$$

In the sequel, when considering the formal expansion of the Ricci tensor in powers of λ, we refer to all the terms multiplied by a certain power λ^k as the λ^k *level*. Moreover, "solving the λ^k level" will mean obtaining $R^{(k)}_{\mu\nu} = 0$.

In the rest of this section and for the sake of clarity we drop the index λ and only write g and $\mathfrak{h}$ instead of g_λ and $\mathfrak{h}_\lambda$.

3.2.1.1 The quasi-linearity

We start by studying the quasi-linear part of the Ricci tensor, also called the wave part in the sequel, that is $-\tilde{\Box}_g g_{\alpha\beta}$. We simply compute

$$\begin{aligned} \tilde{\Box}_g \left(\mathrm{T}\left(\frac{u_0}{\lambda}\right) f \right) = \mathrm{T}\left(\frac{u_0}{\lambda}\right) \tilde{\Box}_g f + \frac{1}{\lambda} \mathrm{T}'\left(\frac{u_0}{\lambda}\right) \left[2g^{\mu\nu}\partial_\mu u_0 \partial_\nu f + (\tilde{\Box}_g u_0) f \right] \\ + \frac{1}{\lambda^2} \mathrm{T}''\left(\frac{u_0}{\lambda}\right) g^{-1}(\mathrm{d}u_0, \mathrm{d}u_0) f \end{aligned} \tag{3.2.5}$$

for any scalar function f and T a trigonometric function. Since $g_0^{-1}(\mathrm{d}u_0, \mathrm{d}u_0) = 0$ we have schematically

$$g^{-1}(\mathrm{d}u_0, \mathrm{d}u_0) = \lambda g^{(1)}_{L_0 L_0} + \lambda^2 g^{(2)}_{L_0 L_0} + O\left(\lambda^3\right). \tag{3.2.6}$$

Therefore, the contribution to $R_{\mu\nu}(g_\lambda)$ of each $\tilde{\Box}_g \lambda^i g^{(i)}$ reads schematically

$$\lambda^0 \left(L_0 \partial_\theta g^{(1)} + g^{(1)}_{L_0 L_0} \partial^2_\theta g^{(1)} \right) + \lambda^1 \left(L_0 \partial_\theta g^{(2)} + g^{(1)}_{L_0 L_0} \partial^2_\theta g^{(2)} + g^{(2)}_{L_0 L_0} \partial^2_\theta g^{(1)} \right) \tag{3.2.7}$$

where we neglect for now $\mathfrak{h}_\lambda$. We recover here the main result of the geometric optics approximation: the waves $g^{(i)}$ solve transport equations along the rays of the optical function u_0 (recall (3.1.5)). These equations will absorb all the terms oscillating as $g^{(i)}$. To make things clearer, we define the notion of *admissible* frequencies:

Definition 3.2.1. *Let $k = 0, 1$. The* ***admissible frequencies*** *at the λ^k level are $\ell\theta$ for $\ell \in [\![1, k+1]\!]$. A non-admissible frequency at a certain level is said to be forbidden at this level.*

Looking at (3.2.7) and at the oscillating behaviour of each $g^{(i)}$ described above, we see that the transport equations for the $g^{(i)}$ will precisely absorb the admissible frequencies.

The quasi-linearity of the Ricci tensor manifests itself by the presence of the eikonal terms $g^{(i)}_{L_0L_0}$. More precisely, if $k = 0, 1$, it produces at the λ^k level the terms $g^{(i)}_{L_0L_0}\partial^2_\theta g^{(k+2-i)}$, which in turn produces $(k+2)\theta$ oscillations if one consider only the top frequency of each $g^{(i)}$ (θ for $g^{(1)}$ and 2θ for $g^{(2)}$). Looking at Definition 3.2.1, these are forbidden frequencies and we need to use the structures of the gauge part and the quadratic part of the Ricci tensor to remove them, see the end of Section 3.2.1.3.

The remainder $\mathfrak{h}_\lambda$ in (3.2.3) being non-oscillating, it simply produces $\tilde{\Box}_g\mathfrak{h}_\lambda$ at the λ^2 level. Therefore, $\mathfrak{h}_\lambda$ will solve a wave equation absorbing any frequency appearing at the λ^2 level, which explains why there is no need to write down this level in (3.2.7), and why $g^{(3)}$ won't be obtained by solving a transport equation.

3.2.1.2 The gauge part

In the last section, we saw how the second order derivatives of the wave part do not produce a loss of two powers of λ since the eikonal term $g^{-1}(\mathrm{d}u_0, \mathrm{d}u_0)$ is of order λ. However, second order derivatives of the metric also appear in the gauge part of the Ricci tensor, that is the one involving first order derivatives of H^ρ whose expression is given by

$$H^\rho = g^{\mu\nu}g^{\rho\sigma}\left(\partial_\mu g_{\sigma\nu} - \frac{1}{2}\partial_\sigma g_{\mu\nu}\right).$$

Let us compute the action of this term on the waves $g^{(i)}$ as we did in the previous section for the wave part:

$$g^{\mu\nu}g^{\rho\sigma}\left(\partial_\mu g_{\sigma\nu} - \frac{1}{2}\partial_\sigma g_{\mu\nu}\right) = \frac{1}{\lambda}g_0^{\mu\nu}g_0^{\rho\sigma}\left(\partial_\mu u_0\partial_\theta g^{(1)}_{\sigma\nu} - \frac{1}{2}\partial_\sigma u_0\partial_\theta g^{(1)}_{\mu\nu}\right) + O(1).$$

The leading term in the previous expression will be of great importance in our construction, it corresponds to the polarization tensor already identified in [CB69].

Definition 3.2.2. *Let S a symmetric 2-tensor, we define its polarization tensor by*

$$\mathrm{Pol}_\sigma(S) = g_0^{\mu\nu}\left(\partial_\mu u_0 S_{\sigma\nu} - \frac{1}{2}\partial_\sigma u_0 S_{\mu\nu}\right). \quad (3.2.8)$$

Its components in the background null frame (see Section 3.1.2) are

$$\mathrm{Pol}_{L_0}(S) = -S_{L_0L_0}, \quad (3.2.9)$$
$$\mathrm{Pol}_A(S) = -S_{AL_0}, \quad (3.2.10)$$
$$\mathrm{Pol}_{\underline{L}_0}(S) = -\delta^{AB}S_{AB}. \quad (3.2.11)$$

Since the main term in the gauge part of the Ricci tensor is $g_{\rho(\alpha}\partial_{\beta)}H^\rho$, the main contributions of the waves $g^{(i)}$ to $R_{\mu\nu}(g_\lambda)$ read schematically

$$\frac{1}{\lambda}\partial_{(\alpha}u_0\mathrm{Pol}_{\beta)}\left(\partial^2_\theta g^{(1)}\right) + \lambda^0\partial_{(\alpha}u_0\mathrm{Pol}_{\beta)}\left(\partial^2_\theta g^{(2)}\right) + \lambda^1\partial_{(\alpha}u_0\mathrm{Pol}_{\beta)}\left(\partial^2_\theta g^{(3)}\right) \quad (3.2.12)$$

Since the quadratic non-linearity only involves first order derivatives, (3.2.12) already implies that solving the λ^{-1} level of the Ricci tensor implies $\mathrm{Pol}\left(g^{(1)}\right) = 0$. Together with (3.2.9) this implies that

$$g^{(1)}_{L_0L_0} = 0 \quad (3.2.13)$$

which remove some of the quasi-linear forbidden frequencies in (3.2.7). The identity (3.2.13) is called the exceptionality condition by Choquet-Bruhat in [CB69] and is a special feature of the Einstein vacuum equations. It also implies that $g^{(1)}_{L_0 A} = 0$, and we will actually prove that $g^{(1)}_{L_0\alpha} = 0$.

Let us now try to understand the role of $\mathrm{Pol}\left(g^{(i)}\right)$ for $i \geq 2$. We introduce the following notions to emphasize the structure at stake:

Definition 3.2.3. *Let S be a symmetric 2-tensor and recall (3.1.13).*

1. *We say that S is **non-tangential** if $S_{XY} = 0$ for all $X, Y \in \mathcal{T}_0$. In coordinates this is equivalent to the existence of a 1-form Q such that $S_{\alpha\beta} = \partial_{(\alpha} u_0 Q_{\beta)}$.*
2. *We say that S is **fully non-tangential** if the only non-zero null components of S is $S_{\underline{L}_0\underline{L}_0}$. In coordinates this is equivalent to the existence of a scalar function f such that $S_{\alpha\beta} = f\partial_\alpha u_0 \partial_\beta u_0$.*

We see in (3.2.12) that the polarization of the waves $g^{(i)}$ only manifest themselves as non-tangential terms in the Ricci tensor, at one level lower than the transport equations that $g^{(i)}$ satisfy. In terms of oscillations, they add the forbidden frequency $(k+2)\theta$ at the λ^k level, for $k = 0, 1$. Therefore, the gauge part in the Ricci tensor can remove some forbidden frequencies with polarization conditions of the form

$$\mathrm{Pol}\left(g^{(i)}\right) = \text{lower order terms}$$

if and only if the problematic terms are non-tangential in the Ricci tensor. This holds for the forbidden frequencies produced by the quadratic non-linearity (see next section) but doesn't hold for the quasi-linear forbidden frequencies in (3.2.7).

The computation (3.2.12) also shows the role of $g^{(3)}$. As explained above, it won't solve a transport equation but will only be useful to absorb forbidden frequency at the λ^1 level.

3.2.1.3 The weak polarized null condition

The semi-linear terms $P_{\alpha\beta}$ (short for $P_{\alpha\beta}(g)(\partial g, \partial g)$) are crucial to our construction since they are the source of backreaction. Indeed we have schematically

$$P_{\alpha\beta} = (\partial g_0)^2 + \left(\partial u_0 \partial_\theta g^{(1)}\right)^2 + \partial g_0 \partial u_0 \partial_\theta g^{(1)} + O(\lambda). \tag{3.2.14}$$

The term $(\partial g_0)^2$ is part of the background equation, while $\partial g_0 \partial u_0 \partial_\theta g^{(1)}$ corresponds to an admissible frequency and thus can be absorbed by the transport equation for $g^{(1)}$. The middle term in (3.2.14) is the most important since it contains a non-oscillating term and an harmonic 2θ, that is a forbidden frequency at the λ^0 level. The non-oscillating term produces the backreaction as it is absorbed by the background equation, which thus becomes an Einstein-type equation with a RHS (see (3.1.2)). Without any additional structure, the harmonic 2θ would be absorbed by the transport equation for $g^{(1)}$, which would therefore be non-linear. However, $P_{\alpha\beta}$ does enjoy additional structure, which we call the *weak polarized null condition* and present now.

The history of null conditions goes back to the work of Klainerman in [Kla86] on the global existence of solutions for semi-linear quadratic wave equations in 3+1 dimensions.

In order to define the weak polarized null condition, we consider the general quadratic form $Q_{\alpha\beta}(T,S)$ acting on symmetric 2-tensors defined by

$$Q_{\alpha\beta}(T,S) = g_0^{\mu\rho} g_0^{\nu\sigma} \partial_{(\alpha} u_0 T_{\rho\sigma} \partial_\mu u_0 S_{\beta)\nu} - \frac{1}{2} g_0^{\mu\rho} g_0^{\nu\sigma} \partial_\alpha u_0 T_{\rho\sigma} \partial_\beta u_0 S_{\mu\nu} \\ - g_0^{\mu\rho} g_0^{\nu\sigma} \partial_\rho u_0 T_{\alpha\nu} \partial_\sigma u_0 S_{\beta\mu} + g_0^{\mu\rho} g_0^{\nu\sigma} \partial_\rho u_0 T_{\sigma\alpha} \partial_\mu u_0 S_{\nu\beta}$$

It corresponds to the leading term in λ when the derivatives $\partial g \partial g$ in $P_{\alpha\beta}$ both hit oscillating parts of the waves $g^{(i)}$ and where T and S denote here the amplitude of the waves. The discussion of Section 3.2.1.2 shows that each wave $g^{(i)}$ will satisfy polarization conditions through their polarization tensor $\mathrm{Pol}\left(g^{(i)}\right)$. We can show that

$$Q_{\alpha\beta}(T,T) = \mathcal{E}\left(T,T\right) \partial_\alpha u_0 \partial_\beta u_0 + g_0^{\nu\sigma} \mathrm{Pol}_\sigma(T) \partial_{(\alpha} u_0 T_{\beta)\nu} - \mathrm{Pol}_\alpha(T) \mathrm{Pol}_\beta(T) \tag{3.2.15}$$

where we define the following energy

$$\mathcal{E}\left(T,S\right) = \frac{1}{4}\left(\mathrm{tr}_{g_0} T\right)\left(\mathrm{tr}_{g_0} S\right) - \frac{1}{2}\left|T\cdot S\right|_{g_0}. \tag{3.2.16}$$

Let us explain our classification of null conditions.

- The standard null condition introduced by Klainerman in [Chr86] and [Kla86] would imply that $Q_{\alpha\beta}(T,T) = 0$, since L_0 is a null vector field.
- In [CB00], Choquet-Bruhat introduces the polarized null condition, which would imply

$$\mathrm{Pol}(T) = 0 \Longrightarrow Q_{\alpha\beta}(T,T) = 0.$$

- We now define the ***weak polarized null condition*** by

$$\mathrm{Pol}(T) = 0 \Longrightarrow Q_{\alpha\beta}(T,T) \text{ is fully non-tangential.}$$

As the computation (3.2.15) shows, the quadratic non-linearity in the Ricci tensor does satisfy the weak polarized null condition. Note the analogy with the weak null condition of Lindblad and Rodnianski, since the term $\mathcal{E}\left(T,T\right) \partial_\alpha u_0 \partial_\beta u_0$ precisely corresponds to their problematic term $P(\partial_\mu h, \partial_\nu h)$, see Proposition 3.1 in [LR10]. This explains our choice of designation.

The weak polarized null condition is the key structure allowing us to deal with the forbidden frequencies caused by the self-interaction of $g^{(1)}$. Indeed, the self-interaction of $g^{(1)}$ produces only fully non-tangential term since $\mathrm{Pol}\left(g^{(1)}\right) = 0$. As explained above, this interaction produces the 2θ forbidden frequency at the λ^0 level (see (3.2.14)), but thanks to the fully non-tangential structure it can be absorbed by the polarization tensor of $g^{(2)}$ which thus satisfies

$$\mathrm{Pol}_\beta\left(g^{(2)}\right) = \frac{1}{2}\mathcal{E}\left(\partial_\theta g^{(1)}, \partial_\theta g^{(1)}\right) \partial_\beta u_0. \tag{3.2.17}$$

In particular this implies

$$\mathrm{Pol}_{\mathcal{T}_0}\left(g^{(2)}\right) = 0. \tag{3.2.18}$$

This last property (together with (3.2.9)) implies that $g^{(2)}_{L_0 L_0} = 0$ (if we only consider the top frequency) which remove the 3θ forbidden frequency at the λ^1 level coming from the quasi-linearity (see (3.2.7)).

We summarize the discussion of Sections 3.2.1.1-3.2.1.2-3.2.1.3 since it describes the key mechanism of our result. Let $k = -1, 0, 1$. The admissible frequencies are absorbed by the transport equation of $g^{(k+1)}$ (for $k \geq 0$). The quasi-linear terms produce forbidden frequencies which rewrite as polarization terms while the semi-linear terms produce forbidden frequencies with the fully non-tangential structure. This allows the polarization tensors of $g^{(k+2)}$ originating from the non-diagonal part of the Ricci tensor to absorb the latter. For this last step, the non-tangential structure would be enough but the *fully* non-tangential structure implies in addition that the quasi-linear forbidden frequencies actually vanish. In particular, this shows how a non-linear theory such as general relativity still implies propagation without deformation since the transport equation for $g^{(1)}$ is linear. This is called the *transparency* phenomenon in the geometric optics literature, see [Mé09].

3.2.1.4 Propagation of the polarization conditions

As explained above, we need the waves $g^{(1)}$ and $g^{(2)}$ to satisfy both transport equations along the rays and polarization conditions. In [CB69], Choquet-Bruhat proves that the polarization condition for $g^{(1)}$, that is $\mathrm{Pol}\left(g^{(1)}\right) = 0$, is propagated by the transport equation $g^{(1)}$ satisfies, which schematically writes $L_0 g^{(1)} = 0$. She shows that if $\mathrm{Pol}\left(g^{(1)}\right) = 0$ holds on Σ_0, and if $g^{(1)}$ solves $L_0 g^{(1)} = 0$, then $\mathrm{Pol}\left(g^{(1)}\right) = 0$ holds on the whole spacetime. This is proved by commuting L_0 and Pol which gives an equation of the form

$$L_0 \mathrm{Pol}\left(g^{(1)}\right) = 0.$$

Unfortunately, the transport equations and polarization conditions satisfied by $g^{(2)}$ (see Sections 3.4.2 and 3.4.4.2) are much more complicated than the one $g^{(1)}$ satisfies and the previous strategy seems unfeasible. Instead of deriving the equations for $\mathrm{Pol}\left(g^{(2)}\right)$ directly from the equations that $g^{(2)}$ satisfies, we treat the polarization conditions for $g^{(2)}$ as gauge conditions. This means that we first solve the hierarchy of equations as if they were satisfied, and then derive the desired transport equations from the contracted Bianchi identities, which states that the Einstein tensor of a Lorentzian metric is divergence free. This shows how we recover the polarization conditions on the whole spacetime, since our choice of initial data will be such that they hold initially (see Corollary 3.4.1).

Therefore, we are able to obtain both the transport equations and the polarization conditions for all the waves in g, proving that the latter is a solution of (3.1.1). It remains to explain how we solve the hierarchy of equations and the challenges its resolution pose from an analytical point of view.

3.2.2 Solving the hierarchy of equations

In this section, we explain how we solve the hierarchy of equations derived from putting the ansatz (3.2.3) into the Einstein vacuum equations (3.1.1). As explained above, the hierarchy of equations is made of transport equations for $g^{(1)}$ and $g^{(2)}$ and a wave equation for $\mathfrak{h}$.

3.2.2.1 The background transport equations

As explained above, the transport equation for $g^{(1)}$ is linear and reads schematically

$$L_0 \partial_\theta g^{(1)} = 0. \tag{3.2.19}$$

If we only focus on the 2θ oscillation in $g^{(2)}$, the second transport equation reads schematically

$$L_0\partial_\theta g^{(2)} = \partial^2 g^{(1)} \tag{3.2.20}$$

where in the RHS of (3.2.20) we only consider the worst term from the perspective of regularity. The system (3.2.19)-(3.2.20) is triangular and solving it causes no problem. In the sequel, the exact form of this system will be called the *background system* because the regularity of its solution only depends on the background spacetime and does not require any bootstrap.

Remark 3.2.1. *The threshold for the background regularity, i.e $N \geq 10$ (see Section 3.1.2), is chosen so that all these background quantities and their derivatives can be ultimately bounded in L^∞. Let us explain briefly why the value 10 fills this requirement.*

If the background metric coefficients are in H^{N+1} (we neglect the weights at spacelike infinity in this discussion), then (3.2.19) and (3.2.20) imply that $g^{(1)} \in H^N$ and $g^{(2)} \in H^{N-2}$. Second derivatives of $g^{(2)}$ will appear as a source term in the wave equation for the remainder $\mathfrak{h}$. In order to prove local well-posedness for this equation we will differenciate it 4 times (see (3.2.29) below) and our worse term is thus $\partial^6 g^{(2)}$. We want to bound it in L^∞ so the usual Sobolev embedding $H^2 \hookrightarrow L^\infty$ implies that we need $N-2 \geq 8$.

3.2.2.2 The quasi-linear coupling and the loss of derivatives

Let us now look at what we call the *reduced system*, that is the system solved by the θ oscillation in $g^{(2)}$ and $\mathfrak{h}$. For a semi-linear equation like the one considered in Chapter 4 of this thesis, this system would also be perfectly triangular. However, due to the quasi-linear terms, our reduced system is not triangular.

As announced, we focus only on the θ oscillation in $g^{(2)}$ and write $g^{(2)}_{\alpha\beta} = \sin\left(\frac{u_0}{\lambda}\right)\mathfrak{F}_{\alpha\beta}$. Considering now the term proportionnal to $\mathfrak{h}$ in the inverse of g that we neglected in the eikonal term (3.2.6), the term $\tilde{\Box}\left(\lambda g^{(1)}\right)$ produces a term proportional to $\lambda\cos\left(\frac{u_0}{\lambda}\right)\mathfrak{h}_{L_0L_0}F^{(1)}$. Moreover, the wave part of the Ricci also contains terms of the form

$$\lambda\cos\left(\frac{u_0}{\lambda}\right)L_0\mathfrak{F} + \lambda^2\left(\Box_g\mathfrak{h} + \sin\left(\frac{u_0}{\lambda}\right)\Box_g\mathfrak{F}\right).$$

Therefore, in order to solve the λ^1 and λ^2 levels we need to solve the following coupled system

$$L_0\mathfrak{F} = \mathfrak{h}, \tag{3.2.21}$$

$$\Box_g\mathfrak{h} = \sin\left(\frac{u_0}{\lambda}\right)\Box_g\mathfrak{F}, \tag{3.2.22}$$

where we only write the terms relevant to the present discussion. Performing energy estimates for (3.2.21) implies schematically that $\partial\mathfrak{F} \sim \partial\mathfrak{h}$ while energy estimates for (3.2.22) give $\partial\mathfrak{h} \sim \partial^2\mathfrak{F}$. Therefore, there is an *a priori* loss of derivatives in the coupling between (3.2.21) and (3.2.22). Regaining this derivatives is the main analytical challenge of our work.

3.2.2.3 Regaining the missing derivative

The main idea is to show that $\Box_g\mathfrak{F}$ is better than $\partial^2\mathfrak{F}$, i.e better than any second order derivatives of $\mathfrak{F}$. To show this, we would like to derive a transport operator for $\Box_g\mathfrak{F}$ from (3.2.21). However we can't hope for any structural property of $[L_0, \Box_g]$ since L_0 is a null vector for g_0 and not g. One option would be to construct the full null foliation associated to g, that is consider $[L, \Box_g]$ for L the spacetime gradient of an optical function u solving $g^{-1}(\mathrm{d}u, \mathrm{d}u) = 0$. This would require to consider the null structure equations and the Bianchi equations in 3+1 as part as our bootstrap argument. To avoid this and to stick to the background null foliation, we benefit from the expansion defining $g - g_0$ and write schematically

$$\Box_g \mathfrak{F} = \Box_{g_0} \mathfrak{F} + \lambda \partial^2 \mathfrak{F}. \tag{3.2.23}$$

We improve the first term in (3.2.23) by considering the commutator $[L_0, \Box_{g_0}]$ for which we can now hope for structural properties. Formally we have from (3.2.21)

$$L_0 \left(\Box_{g_0} \mathfrak{F} \right) = [L_0, \Box_{g_0}] \mathfrak{F} + \Box_{g_0} \mathfrak{h}. \tag{3.2.24}$$

We need to know what are the second derivatives appearing in the commutator $[L_0, \Box_{g_0}]$. This is the content of the following key lemma.

Lemma 3.2.1. *Let f be compactly supported, we have*

$$\left\| [L_0, \Box_{g_0}] f \right\|_{L^2} \leq C(C_0) \left(\| \partial L_0 f \|_{L^2} + \| \Box_{g_0} f \|_{L^2} + \| \partial f \|_{L^2} \right). \tag{3.2.25}$$

Moreover, if $r \geq 1$ we have

$$\left\| \nabla^r [L_0, \Box_{g_0}] f \right\|_{L^2} \leq C(C_0) \left(\| \nabla^r \partial L_0 f \|_{L^2} + \| \nabla^r \Box_{g_0} f \|_{L^2} + \| \partial f \|_{H^r} + \left\| \partial^2 f \right\|_{H^{r-1}} \right). \tag{3.2.26}$$

The proof of this lemma is postponed to Appendix 3.A and is based on the null structure induced by the optical function u_0. In the case of equation (3.2.24), the term $\Box_{g_0} \mathfrak{F}$ from (3.2.25) is treated with Gronwall's inequality and the term $\partial L_0 \mathfrak{F}$ rewrites as $\partial \mathfrak{h}$ thanks to (3.2.21). By using again the expansion for g, we can use equation (3.2.22) for the term $\Box_{g_0} \mathfrak{h}$ in (3.2.24). Therefore, we are able to show that $\Box_{g_0} \mathfrak{F}$ is at the level of $\partial \mathfrak{h}$, which is consistent with (3.2.22). This is done in details in Section 3.6.4.1.

It remains to improve the second term in (3.2.23). As our notation suggests it, we don't use any structure of the second order derivatives involved but rather benefit from the small factor λ in front. The idea is to cancel one derivative with this factor, which rigorously involves a spatial Fourier projector $\Pi_{\leq}$, that is a Fourier multiplier supported in $\left\{ |\xi| \leq \frac{1}{\lambda} \right\}$. If we also define $\Pi_{\geq} = \mathrm{Id} - \Pi_{\leq}$, we have indeed Bernstein-like estimates in L^2:

$$\lambda \partial \Pi_{\leq}(f) \sim f \quad \text{and} \quad \frac{1}{\lambda} \Pi_{\geq}(f) \sim \partial f.$$

The proof of these estimates and the exact definition of the projectors $\Pi_{\leq}$ and $\Pi_{\geq}$ are postponed to Appendix 3.B.3. In order to benefit from those projectors, we need to introduce them into the system (3.2.21)-(3.2.22). Instead of this system, we thus solve

$$L_0 \mathfrak{F} = \Pi_{\leq}(\mathfrak{h}), \tag{3.2.27}$$

$$\Box_g \mathfrak{h} = \sin\left(\frac{u_0}{\lambda} \right) \Box_g \mathfrak{F} + \frac{1}{\lambda} \Pi_{\geq}(\mathfrak{h}). \tag{3.2.28}$$

Using the two Bernstein-like estimates, equation (3.2.27) now implies that $\lambda \partial^2 \mathfrak{F} \sim \partial \mathfrak{h}$ while the extra term in (3.2.28) is absorbed by the energy estimate for the wave operator. This modification of the system is only possible because (3.2.21)-(3.2.22) are obtained as part of the high-frequency expansion of the Ricci tensor. This allows us to artificially remove $\Pi_{\geq}(\mathfrak{h})$ from the transport equation for $\mathfrak{F}$ by simply rewriting the $\lambda \mathfrak{h}$ term as

$$\lambda \Pi_{\leq}(\mathfrak{h}) + \lambda^2 \times \frac{1}{\lambda} \Pi_{\geq}(\mathfrak{h}).$$

This shows how one can use (3.2.23) to gain a derivative, solve (3.2.21)-(3.2.22) and obtain

$$\|\partial\nabla^r\mathfrak{h}\|_{L^2}+\|\partial\nabla^r\mathfrak{F}\|_{L^2}\lesssim\frac{1}{\lambda^r}\tag{3.2.29}$$

for $r\in[\![0,4]\!]$ (we omitted the weight at spacelike infinity for $\mathfrak{h}$). Note that since we modified the equation for $\mathfrak{F}_\lambda$, we need the following commutator estimate to deal with the first term in (3.2.23):

Lemma 3.2.2. *Let u and v be two tempered distribution. We have*

$$\|[u,\Pi_{\leq}]\,\nabla v\|_{L^2}\lesssim\left(\|\nabla u\|_{L^\infty}+\lambda\|u\|_{H^{\frac{7}{2}}}\right)\|v\|_{L^2}\,.\tag{3.2.30}$$

This lemma will be used to estimate $[\Pi_{\leq},\Box_{g_0}]$. Its proof is postponed to Appendix 3.B.3 and relies on the Littlewood-Paley decomposition, and we introduce in Section 3.B.1 the material needed.

3.2.2.4 The order of precision of the ansatz

Finally, let us comment on the order of precision of the high-frequency ansatz defining g_λ. As (3.2.3) shows, the order of precision is 2, since the non-oscillating remainder $\mathfrak{h}_\lambda$ appears first at the order λ^2 in g_λ. This is similar to the construction of high-frequency ansatz in [HL18b] in $\mathbb{U}(1)$ symmetry, but different from the results of Chapter 4, where we are able to construct high-frequency ansatz of arbitrary precision for a toy model.

As in Chapter 4, the value of this order of precision is linked to the method of proof we use to solve the equations at stake. To make this clearer, let us assume that this value, denoted by $K\in\mathbb{Z}$, is not fixed yet.

The transport equations only involve the transport operator L_0 which is part of the background and therefore their resolution is independent from λ. However, we use energy estimates associated to g_λ to solve (3.2.28). As shown in [Sog95] (see also Lemma 3.6.9 in this chapter) this energy estimates reads schematically

$$\|\partial u\|_{L^2}(t)\lesssim\exp\left(\int_0^t\|\partial g_\lambda\|_{L^\infty}\right)\left(\|\partial u\|_{L^2}(0)+\int_0^t\left\|\tilde{\Box}_{g_\lambda}u\right\|_{L^2}\right).\tag{3.2.31}$$

Therefore, in order to construct a solution to (3.2.28) with a time of existence independent of λ, we need $\|\partial g_\lambda\|_{L^\infty}$ to be bounded uniformly in the high-frequency limit $\lambda\longrightarrow0$, since it appears as coefficient in (3.2.31). If we focus on the term $\lambda^K\mathfrak{h}_\lambda$ in g_λ, the usual Sobolev embedding $H^2\hookrightarrow L^\infty$ gives

$$\left\|\lambda^K\partial\mathfrak{h}_\lambda\right\|_{L^\infty}\lesssim\lambda^K\|\partial\mathfrak{h}_\lambda\|_{H^2}\lesssim\lambda^{K-2}$$

where we used the fact that one loses one power of λ for each derivatives of $\partial\mathfrak{h}_\lambda$ (see (3.2.29)). See Lemma 3.6.1 for a more rigorous proof. We see why we need K to be at least equal to 2. The present chapter then shows that $K=2$ is sufficient to construct local high-frequency solutions to (3.1.1) on $[0,1]\times\mathbb{R}^3$.

An interesting problem would be to construct similar solutions but of greater order of precision, which could pave the way for the study of the long time dynamics of high-frequency waves. Indeed, Chapter 4 of this thesis shows with a toy model how the vector field method requires $K\geq4$.

3.2.3 Outline of the chapter

We give here an outline of the Chapter's remainder which is concerned with the proof of Theorem 3.1.2 and follows the strategy depicted in the previous sections.

- In Section 3.3, we give the final expression of the ansatz for g_λ and expand its Ricci tensor.
- In Section 3.4, we state the hierarchy of equations as well as the polarization and generalised wave gauge conditions. The hierarchy is composed of the background system and the reduced system. We also define the initial data.
- In Section 3.5, we solve the background system.
- In Section 3.6, we solve the reduced system.
- In Section 3.7, we show that the metric g_λ is actually a solution of (3.1.1) by propagating the polarization and generalised wave gauge conditions.

3.3 The high-frequency ansatz

In this section, we present the ansatz for the spacetime metric g_λ and compute its Ricci tensor.

3.3.1 High-frequency expansion of the metric

The metric we consider is given by

$$g_\lambda = g_0 + \lambda g^{(1)} + \lambda^2 \left(g^{(2)} + \mathfrak{h}_\lambda \right) + \lambda^3 g^{(3)} \tag{3.3.1}$$

where

$$\begin{aligned} g^{(1)} &= F^{(1)} \cos\left(\frac{u_0}{\lambda}\right), \\ g^{(2)} &= \mathfrak{F}_\lambda \sin\left(\frac{u_0}{\lambda}\right) + F^{(2,1)} \sin\left(\frac{u_0}{\lambda}\right) + F^{(2,2)} \cos\left(\frac{2u_0}{\lambda}\right). \end{aligned}$$

The symmetric 2-tensors $F^{(1)}$ and $F^{(2,i)}$ for $i = 1, 2$ are called the background perturbations. They satisfy the background system, defined in the Section 3.4.2. The symmetric 2-tensors $\mathfrak{F}_\lambda$ and $\mathfrak{h}_\lambda$ satisfy the reduced system, defined in Section 3.4.3. We use a different font to emphasize their special role. The exact definition of $g^{(3)}$ will be given in Section 3.4.5, let us just say for now that it is a polynomial in terms of the background perturbations and their derivatives and is independent $\mathfrak{h}_\lambda$ but does depends on $\mathfrak{F}_\lambda$ without derivatives. The background perturbations tensors don't depend on λ, unlike the tensors $\mathfrak{F}_\lambda$, $\mathfrak{h}_\lambda$ and the metric g_λ itself. However, for the sake of clarity we drop from now on this notation and only write $\mathfrak{F}$, $\mathfrak{h}$ and g.

Remark 3.3.1. *The presence of $\mathfrak{F}$ and $F^{(2,1)}$ in $g^{(2)}$ seems redundant and is not necessary. However it seems useful to the author to distinguish between the part of the θ oscillation in $g^{(2)}$ coupled to $\mathfrak{h}$ (see Sections 3.2.2.2 for a informal discussion) and the part whose only purpose is to absorb terms coming from $g^{(1)}$ such as $\Box_{g_0} F^{(1)}$ and which can thus be treated as a background quantity.*

We can formally compute the high-frequency expansion of the inverse of g. In coordinates it is given by

$$g^{\mu\nu} = g_0^{\mu\nu} + \lambda (g^{\mu\nu})^{(1)} + \lambda^2 (g^{\mu\nu})^{(2)} + \lambda^3 H^{\mu\nu} \tag{3.3.2}$$

where we define

$$(g^{\mu\nu})^{(1)} = -(F^{(1)})^{\mu\nu} \cos\left(\frac{u_0}{\lambda}\right), \tag{3.3.3}$$

$$\begin{aligned}(g^{\mu\nu})^{(2)} &= g_0^{\mu\rho}(F^{(1)})^{\nu\sigma} F^{(1)}_{\rho\sigma} \cos^2\left(\frac{u_0}{\lambda}\right) - \mathfrak{F}^{\mu\nu} \sin\left(\frac{u_0}{\lambda}\right) - \mathfrak{h}^{\mu\nu} \\ &\quad - (F^{(2,1)})^{\mu\nu} \sin\left(\frac{u_0}{\lambda}\right) - (F^{(2,2)})^{\mu\nu} \cos\left(\frac{2u_0}{\lambda}\right)\end{aligned} \tag{3.3.4}$$

On the RHS of those expressions, all the inverse are taken with respect of the background metric g_0. For example we have $(F^{(1)})^{\mu\nu} = g_0^{\mu\rho} g_0^{\nu\sigma} F^{(1)}_{\rho\sigma}$. The quantity $H^{\mu\nu}$ is defined so that (3.3.2) holds with $(g^{\mu\nu})^{(i)}$ for $i = 1, 2$ being defined by (3.3.3)-(3.3.4), and one can verify by hand that $H^{\mu\nu} = O(1)$.

Remark 3.3.2. *Note that we denote by g^{-k} any product of k coefficients of the inverse metric g^{-1}. This also applies to the background inverse metric g_0^{-1}.*

3.3.2 High-frequency expansion of the Ricci tensor

In this section, we compute the Ricci tensor of the metric g given by (3.3.1). In order to lighten the computations, we assume that $\mathrm{Pol}\left(F^{(1)}\right) = 0$, where Pol is the polarization operator defined in (3.2.8). We also assume $F^{(1)}_{L_0\underline{L}_0} = 0$, which altogether gives $F^{(1)}_{L_0\alpha} = 0$ and $\mathrm{tr}_{g_0} F^{(1)} = 0$. The fact that $F^{(1)}$ satisfy these conditions will be proved in Section 3.5.3.1.

The expansion of the Ricci tensor is based on (3.2.1), i.e its decomposition in generalised wave coordinates, and we first decompose the wave part $\tilde{\Box}_g g_{\alpha\beta}$, then the gauge part H^ρ and finally the quadratic non-linearity $P_{\alpha\beta}$. We recall the notion of admissible/forbidden frequencies (see Definition 3.2.1): at the λ^k level, only the frequencies $\ell\theta$ for $\ell \in [\![1, k+1]\!]$ are admissible, with $k = 0, 1$.

The exact expression of the expansions are complicated and we proceed as follows to present them in the clearest way possible. We start by giving the formal expansions in powers of λ (see (3.3.5), (3.3.17) and (3.3.27)), and then give the oscillating expansion of each coefficients in Lemmas 3.3.1, 3.3.2 and 3.3.3 for the wave part, Lemmas 3.3.4, 3.3.5 and 3.3.6 for the gauge part and Lemmas 3.3.7 and 3.3.8 for the quadratic non-linearity.

3.3.2.1 The wave part

We start by the expansion of the wave part of the Ricci tensor of g. Formally we have

$$\tilde{\Box}_g g_{\alpha\beta} = W^{(0)}_{\alpha\beta} + \lambda W^{(1)}_{\alpha\beta} + \lambda^2 W^{(\geq 2)}_{\alpha\beta}. \tag{3.3.5}$$

Lemma 3.3.1 (Expression of $W^{(0)}$)**.** *We have*

$$W^{(0)}_{\alpha\beta} = \tilde{\Box}_{g_0}(g_0)_{\alpha\beta} - \sin\left(\frac{u_0}{\lambda}\right)\left[-2L_0 F^{(1)}_{\alpha\beta} + (\tilde{\Box}_{g_0} u_0) F^{(1)}_{\alpha\beta}\right]. \tag{3.3.6}$$

Proof. The main ingredient is the computation (3.2.5). We can use the expansion of the inverse given in (3.3.2)-(3.3.4) and $F^{(1)}_{L_0\alpha} = 0$ to expand first the eikonal term $g^{-1}(\mathrm{d}u_0, \mathrm{d}u_0)$:

$$\begin{aligned} & g^{-1}(\mathrm{d}u_0, \mathrm{d}u_0) \\ & = g_0^{-1}(\mathrm{d}u_0, \mathrm{d}u_0) - \lambda F^{(1)}_{L_0L_0} \cos\left(\frac{u_0}{\lambda}\right) \\ & \quad - \lambda^2 \left(\mathfrak{F}_{L_0L_0} \sin\left(\frac{u_0}{\lambda}\right) + F^{(2,1)}_{L_0L_0} \sin\left(\frac{u_0}{\lambda}\right) + F^{(2,2)}_{L_0L_0} \cos\left(\frac{2u_0}{\lambda}\right) + \mathfrak{h}_{L_0L_0} \right) + O\left(\lambda^3\right) \\ & = -\lambda^2 \left(\mathfrak{F}_{L_0L_0} \sin\left(\frac{u_0}{\lambda}\right) + F^{(2,1)}_{L_0L_0} \sin\left(\frac{u_0}{\lambda}\right) + F^{(2,2)}_{L_0L_0} \cos\left(\frac{2u_0}{\lambda}\right) + \mathfrak{h}_{L_0L_0} \right) + O\left(\lambda^3\right) \qquad (3.3.7) \end{aligned}$$

where we also used (3.1.3). The absence of a λ^0 term in (3.3.7) explains why there is no λ^{-1} term in the expansion of $\tilde{\Box}_g g_{\alpha\beta}$ given by (3.3.5). Similarly we have

$$g^{\mu\nu}\partial_\mu u_0 \partial_\nu f = -L_0 f + O\left(\lambda^2\right) \qquad (3.3.8)$$

and

$$\tilde{\Box}_g f = \tilde{\Box}_{g_0} f - \lambda \cos\left(\frac{u_0}{\lambda}\right) (F^{(1)})^{\mu\nu} \partial_\mu \partial_\nu f + O\left(\lambda^2\right) \qquad (3.3.9)$$

for f a scalar function. Using the general formula (3.2.5) and the expansions (3.3.7), (3.3.8) and (3.3.9) we can expand $\tilde{\Box}_g g_{\alpha\beta}$ and obtain the different terms in (3.3.5). In particular the λ^0 terms in $\tilde{\Box}_g g_{\alpha\beta}$, i.e $W^{(0)}_{\alpha\beta}$ come from $\partial^2 g_0$ and $\partial u_0 \partial_\theta \partial g^{(1)}$. □

Remark 3.3.3. *Note that the absence of the forbidden frequency 2θ is due to $F^{(1)}_{L_0\alpha} = 0$.*

Lemma 3.3.2 (Expression of $W^{(1)}$)**.** *We have*

$$\begin{aligned} W^{(1)}_{\alpha\beta} = {} & \cos\left(\frac{u_0}{\lambda}\right) W^{(1,1)}_{\alpha\beta} + \sin\left(\frac{2u_0}{\lambda}\right) W^{(1,2)}_{\alpha\beta} \qquad (3.3.10) \\ & + \frac{1}{2} \sin\left(\frac{2u_0}{\lambda}\right) \mathfrak{F}_{L_0L_0} F^{(1)}_{\alpha\beta} + \cos\left(\frac{u_0}{\lambda}\right) \cos\left(\frac{2u_0}{\lambda}\right) F^{(2,2)}_{L_0L_0} F^{(1)}_{\alpha\beta} \end{aligned}$$

where $W^{(1,1)}$ and $W^{(1,2)}$ are given in (3.3.11) and (3.3.12).

Proof. The λ terms in $\tilde{\Box}_g g_{\alpha\beta}$ come from $\partial^2 g^{(1)}$ and $\partial u_0 \partial_\theta \partial g^{(2)}$, which explains the oscillating decomposition of $W^{(1)}$ in (3.3.10). More precisely, the terms $W^{(1,i)}$ are given by

$$\begin{aligned} W^{(1,1)}_{\alpha\beta} = {} & -2L_0 F^{(2,1)}_{\alpha\beta} + (\tilde{\Box}_{g_0} u_0) F^{(2,1)}_{\alpha\beta} + \tilde{W}^{(1,1)}_{\alpha\beta} \qquad (3.3.11) \\ & - 2L_0 \mathfrak{F}_{\alpha\beta} + (\tilde{\Box}_{g_0} u_0) \mathfrak{F}_{\alpha\beta} + \mathfrak{h}_{L_0L_0} F^{(1)}_{\alpha\beta}, \\ W^{(1,2)}_{\alpha\beta} = {} & -2 \left(-2L_0 F^{(2,2)}_{\alpha\beta} + (\tilde{\Box}_{g_0} u_0) F^{(2,2)}_{\alpha\beta} + \tilde{W}^{(1,2)}_{\alpha\beta} - \frac{1}{4} F^{(2,1)}_{L_0L_0} F^{(1)}_{\alpha\beta} \right), \qquad (3.3.12) \end{aligned}$$

with $\tilde{W}^{(1,i)}$ being some irrelevant lower order terms formally admitting the following expression

$$\tilde{W}^{(1,i)} = g_0^{-2} F^{(1)} \partial^2 g_0 + g_0^{-2} \partial^2 F^{(1)} + g_0^{-2} (F^{(1)})^2 \partial^2 u_0. \qquad (3.3.13)$$

□

Remark 3.3.4. *Note the appearance of the eikonal terms $\mathfrak{F}_{L_0L_0}$ and $F^{(2,2)}_{L_0L_0}$ in $W^{(1)}$, which will require a special treatment. The exact expression in front of $F^{(2,2)}_{L_0L_0}$ in (3.3.10) is irrelevant, the only important remark to be made is that it involves the 3θ frequency, which is forbidden at the λ level.*

Lemma 3.3.3 (Expression of $W^{(\geq 2)}$). *We have*

$$W^{(\geq 2)}_{\alpha\beta} = \tilde{\Box}_g \mathfrak{h}_{\alpha\beta} + \sin\left(\frac{u_0}{\lambda}\right)\tilde{\Box}_g \mathfrak{F}_{\alpha\beta} + \lambda\tilde{\Box}_g g^{(3)}_{\alpha\beta} + \tilde{W}^{(\geq 2)}_{\alpha\beta} \tag{3.3.14}$$

where $\tilde{W}^{(\geq 2)}_{\alpha\beta}$ *contains only lower order terms and is given in* (3.3.15).

Proof. Since the λ^2 terms in the Ricci tensor will be absorbed by the wave equation for $\mathfrak{h}$, there is no need for a decomposition in frequencies and (3.3.14) simply follows from considering first all the second derivatives of $\mathfrak{F}$, $\mathfrak{h}$ and $g^{(3)}$ and then putting all the lower order quantities into $\tilde{W}^{(\geq 2)}_{\alpha\beta}$. This gives the following schematic expression:

$$\begin{aligned}\tilde{W}^{(\geq 2)} &= (g^{-1})^{(\geq 3)}(\partial u_0)^2 F^{(1)} + (g^{-1})^{(\geq 2)}\left(\partial^2 g_0 + \partial^2 u_0 + \partial u_0 \partial F^{(1)} + (\partial u_0)^2 F^{(2,i)}\right) \\ &\quad + (g^{-1})^{(\geq 1)}\left(\partial^2 F^{(1)} + \partial^2 u_0 F^{(2,i)} + \partial u_0 \partial F^{(2,i)}\right) + g^{-1}\partial^2 F^{(2,i)}.\end{aligned} \tag{3.3.15}$$

□

As explained in Section 3.2.1.1, the forbidden frequency 3θ at the λ^1 level comes from the eikonal term in (3.2.5). Moreover, it is caused by the "top frequency term", i.e $F^{(2,2)}_{L_0L_0}$. See the end of Section 3.4.4.2 to see how we deal with these quasi-linear forbidden frequencies.

3.3.2.2 The gauge part

In Section 3.2.1.2, we explained how the main part in the gauge term H^ρ are the polarization tensors Pol of each wave in (3.3.1). These tensors will play an important role in the consistency of our ansatz by absorbing forbidden frequencies. The gauge term is also responsible for a potential loss of hyperbolicity. Indeed, the two non-oscillating terms in (3.3.1), i.e g_0 and $\mathfrak{h}$ will solve wave equations, but the gauge term H^ρ contains first order derivatives of those tensors which leads to the loss of hyperbolicity announced (since the Ricci tensor contains derivatives of H^ρ). The generalised wave gauge choice is precisely made to cancel those problematic terms. We already assume the usual wave gauge condition for the background in (3.1.6), and for $\mathfrak{h}_\lambda$ the problematic terms are given by

$$\begin{aligned}\Upsilon^\rho &= g^{\mu\nu}g^{\rho\sigma}\left(\partial_\mu \mathfrak{h}_{\sigma\nu} - \frac{1}{2}\partial_\sigma \mathfrak{h}_{\mu\nu} + \sin\left(\frac{u_0}{\lambda}\right)\left(\partial_\mu \mathfrak{F}_{\sigma\nu} - \frac{1}{2}\partial_\sigma \mathfrak{F}_{\mu\nu}\right) + \lambda\left(\partial_\mu g^{(3)}_{\sigma\nu} - \frac{1}{2}\partial_\sigma g^{(3)}_{\mu\nu}\right)\right) \\ &\quad - g_0^{\rho\sigma}\mathfrak{h}^{\mu\nu}\left(\partial_\mu (g_0)_{\sigma\nu} - \frac{1}{2}\partial_\sigma (g_0)_{\mu\nu} - \sin\left(\frac{u_0}{\lambda}\right)\left(\partial_\mu u_0 F^{(1)}_{\sigma\nu} - \frac{1}{2}\partial_\sigma u_0 F^{(1)}_{\mu\nu}\right)\right)\end{aligned} \tag{3.3.16}$$

Concretely, Υ^ρ contains all derivatives of $\mathfrak{h}$ and $\mathfrak{F}$. Though it doesn't satisfy a wave equation, we also put $\partial\mathfrak{F}$ in (3.3.16), its presence will be justified by the analysis of the hierarchy of equations, which will show that $\partial\mathfrak{F}$ are at the level of $\partial\mathfrak{h}$. Note that in the first line, $\partial g^{(3)}$ is the non-frequential derivative of $g^{(3)}$, i.e we don't differentiate the oscillating parts of $g^{(3)}$. These $\partial g^{(3)}$ terms are here since they involve $\partial\mathfrak{F}$. The presence of the last line in (3.3.16) is linked to $g^{(3)}$ and will be explained at the end of this section.

We now introduce the formal high-frequency expansion of H^ρ:

$$H^\rho = \lambda\left((H^{(1)})^\rho + \cos\left(\frac{u_0}{\lambda}\right) g_0^{\rho\sigma}\mathrm{Pol}_\sigma(\mathfrak{F})\right) + \lambda^2\left((H^{(2)})^\rho + \Upsilon^\rho\right) + \lambda^3 (H^{(\geq 3)})^\rho. \tag{3.3.17}$$

Note that there is no λ^0 term thanks to (3.1.6) and $\mathrm{Pol}\left(F^{(1)}\right)=0$. For clarity, we denote by $\mathring{H}$ the part of H containing only allowed terms from the wave equation perspective, that is

$$\mathring{H}^\rho = H^\rho - \lambda^2 \Upsilon^\rho. \tag{3.3.18}$$

Lemma 3.3.4 (Expression of $H^{(1)}$). *We have*

$$\begin{aligned}(H^{(1)})^\rho = {} & \cos\left(\frac{u_0}{\lambda}\right)\left(g_0^{\rho\sigma}\mathrm{Pol}_\sigma\left(F^{(2,1)}\right) + (\tilde{H}^{(1,1)})^\rho\right) \\ & + \sin\left(\frac{2u_0}{\lambda}\right)\left(-2g_0^{\rho\sigma}\mathrm{Pol}_\sigma\left(F^{(2,2)}\right) + (\tilde{H}^{(1,2)})^\rho\right)\end{aligned} \tag{3.3.19}$$

where $\tilde{H}^{(1,1)}$ and $\tilde{H}^{(1,2)}$ are given in (3.3.20) and (3.3.21).

Proof. We decompose $(H^{(1)})^\rho$ into frequencies and highlight its dependence on $g^{(2)}$. This gives the expression (3.3.19). We only need a formal expression of $\tilde{H}^{(1,1)}$:

$$\tilde{H}^{(1,1)} = g_0^{-2}\left(\partial F^{(1)} + g_0^{-1}F^{(1)}\partial g_0\right). \tag{3.3.20}$$

However, we need the precise expression of $\tilde{H}^{(1,2)}$, wich comes from $g_0^{-1}(g^{-1})^{(1)}\partial u_0 \partial_\theta g^{(1)}$ and is given by

$$(\tilde{H}^{(1,2)})^\rho = -\frac{1}{4}g_0^{\rho\sigma}\partial_\sigma u_0 \left|F^{(1)}\right|^2_{g_0}. \tag{3.3.21}$$

□

Remark 3.3.5. *Note that $\mathfrak{F}$ doesn't appear in $H^{(1)}$ (even though it plays a similar role as $F^{(2,i)}$) since its polarization contribution is already in (3.3.17). As Lemma 3.7.1 will show, this contribution actually vanishes.*

Lemma 3.3.5 (Expression of $H^{(2)}$). *We have*

$$(H^{(2)})^\rho = g_0^{\rho\sigma}\mathrm{Pol}_\sigma\left(\partial_\theta g^{(3)}\right) + (\tilde{H}^{(2)})^\rho \tag{3.3.22}$$

where $\tilde{H}^{(2)}$ is given in (3.3.23).

Proof. Since $H^{(2)}$ will be absorbed at the λ^1 level by $g^{(3)}$, we don't need to specify its decomposition in frequency. The expression (3.3.22) only means that we highlight the dependency of $H^{(2)}$ on $g^{(3)}$, meaning that the quantity $(\tilde{H}^{(2)})^\rho$ is defined so that (3.3.22) holds, and does not depends on $g^{(3)}$. Moreover, thanks to our choice of Υ^ρ (see (3.3.16)), $(\tilde{H}^{(2)})^\rho$ also does not depends on $\mathfrak{h}$. Indeed, $(\tilde{H}^{(2)})^\rho$ could depend on $\mathfrak{h}$ through two types of terms.

- The $\partial\mathfrak{h}$ appearing in $g_0^{-1}(\partial g)^{(2)}$, which corresponds to the first line in (3.3.16). The latter also ensures that $(\tilde{H}^{(2)})^\rho$ does not depend on $\partial\mathfrak{F}$.
- The $\mathfrak{h}$ appearing in $(g^{-1})^{(2)}(\partial g)^{(0)}$, which corresponds to the last line in (3.3.16).

The term $(\tilde{H}^{(2)})^\rho$ does however depend in a polynomial fashion on the quantities g_0, $F^{(1)}$ and $F^{(2,i)}$ and their first derivatives. Therefore, we can write schematically

$$\begin{aligned}\tilde{H}^{(2)} = {} & g_0^{-3}\left((F^{(1)})^2 + F^{(2,i)} + \mathfrak{F}\right)(\partial g_0 + \partial u_0 F^{(1)}) \\ & + g_0^{-3}F^{(1)}(\partial F^{(1)} + \partial u_0 F^{(2,i)} + \partial u_0 \mathfrak{F}) + g_0^{-2}\partial F^{(2,i)}\end{aligned} \tag{3.3.23}$$

where we used a schematic version of (3.3.4), that is

$$(g^{-1})^{(2)} = g_0^{-2}\left((F^{(1)})^2 + F^{(2,i)} + \mathfrak{h} + \mathfrak{F}\right).$$

□

Lemma 3.3.6 (Expression of $H^{(\geq 3)}$). *We have*

$$H^{(\geq 3)} = (g^{-2})^{(\geq 3)}\left(\partial g_0 + \partial u_0 F^{(1)}\right) + (g^{-2})^{(\geq 2)}\left(\partial F^{(1)} + \partial u_0 F^{(2,i)} + \partial u_0 \mathfrak{F}\right) + (g^{-2})^{(\geq 1)}\partial F^{(2,i)}. \tag{3.3.24}$$

Proof. Since $H^{(\geq 3)}$ will only appear at the RHS of the wave equation for the remainder $\mathfrak{h}$, its only important property is its independancy from $\partial\mathfrak{h}$ and $\partial\mathfrak{F}$. This holds thanks to (3.3.16), where in particular we put the $\partial g^{(3)}$ in Υ^ρ since $g^{(3)}$ contains linear terms in $\mathfrak{F}$ (see Section 3.4.5 below). It does depend on g_0, $F^{(1)}$, $F^{(2,i)}$, and their first derivatives, and on $\mathfrak{F}$, $\mathfrak{h}$ and $g^{(3)}$. The terms depending on $g^{(3)}$ in $H^{(\geq 3)}$ are still $O(1)$ with respect to λ, since we lose one λ power when the derivatives hit the oscillating parts of $g^{(3)}$, producing the $\partial_\theta g^{(3)}$ already included in $H^{(2)}$. This justifies the schematic expression (3.3.24). □

The gauge term H^ρ is differenciated in the Ricci tensor, and we now give the first two orders of the high-frequency expansion of these terms. We have

$$\left(H^\rho\partial_\rho g_{\alpha\beta} + g_{\rho(\alpha}\partial_{\beta)}H^\rho\right)^{(0)} = (g_0)_{\rho(\alpha}\partial_{\beta)}u_0\partial_\theta(H^{(1)})^\rho - \sin\left(\frac{u_0}{\lambda}\right)\partial_{(\alpha}u_0\mathrm{Pol}_{\beta)}(\mathfrak{F}), \tag{3.3.25}$$

$$\begin{aligned}\left(H^\rho\partial_\rho g_{\alpha\beta} + g_{\rho(\alpha}\partial_{\beta)}H^\rho\right)^{(1)} &\\ = \left((H^{(1)})^\rho + \cos\left(\frac{u_0}{\lambda}\right)g_0^{\rho\sigma}\mathrm{Pol}_\sigma(\mathfrak{F})\right)&\left(\partial_\rho(g_0)_{\alpha\beta} - \sin\left(\frac{u_0}{\lambda}\right)\partial_\rho u_0 F^{(1)}_{\alpha\beta}\right) \\ &+ (g_0)_{\rho(\alpha}\left(\partial_{\beta)}(H^{(1)})^\rho + \cos\left(\frac{u_0}{\lambda}\right)\partial_{\beta)}\left(g_0^{\rho\sigma}\mathrm{Pol}_\sigma(\mathfrak{F})\right)\right) + (g_0)_{\rho(\alpha}\partial_{\beta)}u_0\partial_\theta\left((H^{(2)})^\rho + \Upsilon^\rho\right) \\ &+ \cos\left(\frac{u_0}{\lambda}\right)F^{(1)}_{\rho(\alpha}\partial_{\beta)}u_0\left(\partial_\theta(H^{(1)})^\rho - \sin\left(\frac{u_0}{\lambda}\right)g_0^{\rho\sigma}\mathrm{Pol}_\sigma(\mathfrak{F})\right),\end{aligned} \tag{3.3.26}$$

where by $\partial_{\beta)}(H^{(1)})^\rho$ in (3.3.26) we denote the derivative of $(H^{(1)})^\rho$ without differenciating its oscillating parts (i.e the $\cos\left(\frac{u_0}{\lambda}\right)$ and $\sin\left(\frac{2u_0}{\lambda}\right)$ in (3.3.19)).

3.3.2.3 The quadratic non-linearity

In this section we expand $P_{\alpha\beta}(g)(\partial g, \partial g)$ (see (3.2.2) for an exact expression). Formally we have

$$P_{\alpha\beta}(g)(\partial g, \partial g) = P^{(0)}_{\alpha\beta} + \lambda P^{(1)}_{\alpha\beta} + \lambda^2 P^{(\geq 2)}_{\alpha\beta}. \tag{3.3.27}$$

The main structural remarks on $P_{\alpha\beta}$ were already made in Section 3.2.1.3, where we defined the weak polarized null condition. We recall the main consequences of this structure: the self-interaction of a wave T produces fully non-tangential terms and $\mathrm{Pol}(T)$ terms (see (3.2.15)).

Lemma 3.3.7 (Expression of $P^{(0)}$). *We have*

$$P^{(0)}_{\alpha\beta} = P^{(0)}_{\alpha\beta}\left[g^{(1)}\right] + P^{(0)}_{\alpha\beta}\left[g_0\right] \tag{3.3.28}$$

where

$$P^{(0)}_{\alpha\beta}\left[g_0\right] = P_{\alpha\beta}(g_0)(\partial g_0, \partial g_0), \tag{3.3.29}$$

$$\begin{aligned}P^{(0)}_{\alpha\beta}\left[g^{(1)}\right] = \frac{1}{2}\mathcal{E}\left(F^{(1)}, F^{(1)}\right)\partial_\alpha u_0\partial_\beta u_0 + \cos\left(\frac{2u_0}{\lambda}\right)\partial_{(\alpha}u_0\hat{P}^{(0,2)}_{\beta)}\left[g^{(1)}\right] \\ - \sin\left(\frac{u_0}{\lambda}\right)\left(-2L_0^\rho\Gamma(g_0)^\nu_{(\alpha\rho}F^{(1)}_{\beta)\nu} + \partial_{(\alpha}u_0\hat{P}^{(0,1)}_{\beta)}\left[g^{(1)}\right]\right),\end{aligned} \tag{3.3.30}$$

and where $\hat{P}^{(0,2)}_\beta\left[g^{(1)}\right]$ and $\hat{P}^{(0,1)}_\beta\left[g^{(1)}\right]$ are given in (3.3.31) and (3.3.32).

Proof. The λ^0 terms in $P_{\alpha\beta}(g)(\partial g, \partial g)$ are of the form $g_0^{-2}(\partial g)^{(0)}(\partial g)^{(0)}$, with $(\partial g)^{(0)}$ being itself of the form $\partial g_0 + \partial u_0 \partial_\theta g^{(1)}$. We decompose first $P^{(0)}$ in terms depending on $g^{(1)}$ which gives (3.3.28).The background dependent terms are simply given by $g_0^{-1} g_0^{-1} \partial g_0 \partial g_0$, that is the quadratic non-linearity appearing on the RHS of (3.1.7), hence (3.3.29). The terms depending on $g^{(1)}$ come from the self-interaction of $g^{(1)}$ and from the interaction between $g^{(1)}$ and the background. Since $\mathrm{Pol}\left(F^{(1)}\right) = 0$ we obtain (3.3.30) where

$$\hat{P}_\beta^{(0,1)}\left[g^{(1)}\right] = g_0^{-2} F^{(1)} \partial g_0. \tag{3.3.31}$$

and

$$\hat{P}_\beta^{(0,2)}\left[g^{(1)}\right] = -\frac{1}{4}\mathcal{E}\left(F^{(1)}, F^{(1)}\right)\partial_\beta u_0. \tag{3.3.32}$$

□

Remark 3.3.6. *The forbidden frequency in (3.3.28) has the non-tangential structure, allowing the polarization tensors of $g^{(2)}$ to absorb it.*

Lemma 3.3.8 (Expression of $P^{(1)}$). *We have*

$$P_{\alpha\beta}^{(1)} = P_{\alpha\beta}^{(1)}\left[g^{(2)}\right] + P_{\alpha\beta}^{(1)}\left[g^{(\leq 1)}\right] \tag{3.3.33}$$

where

$$P_{\alpha\beta}^{(1)}\left[g^{(\leq 1)}\right] = \cos\left(\frac{u_0}{\lambda}\right) P_{\alpha\beta}^{(1,1)} + \sin\left(\frac{2u_0}{\lambda}\right) P_{\alpha\beta}^{(1,2)} + \cos\left(\frac{3u_0}{\lambda}\right) P_{\alpha\beta}^{(1,3)} \tag{3.3.34}$$

$$\begin{aligned} P_{\alpha\beta}^{(1)}\left[g^{(2)}\right] &= -2L_0^\rho \Gamma(g_0)_{(\alpha\rho}^\nu \left(\cos\left(\frac{u_0}{\lambda}\right)\mathfrak{F}_{\nu\beta)} + \cos\left(\frac{u_0}{\lambda}\right) F_{\nu\beta)}^{(2,1)} - 2\sin\left(\frac{2u_0}{\lambda}\right) F_{\nu\beta)}^{(2,2)}\right) \\ &\quad + \partial_{(\alpha} u_0 \hat{P}_{\beta)}^{(1)}\left[g^{(2)}\right] \end{aligned} \tag{3.3.35}$$

and where $P_{\alpha\beta}^{(1,1)}$, $P_{\alpha\beta}^{(1,2)}$ and $P_{\alpha\beta}^{(1,3)}$ are given in (3.3.37) and (3.3.38), and $\hat{P}_\beta^{(1)}\left[g^{(2)}\right]$ is given in (3.3.40) (see also (3.3.41)).

Proof. The λ terms in $P_{\alpha\beta}(g)(\partial g, \partial g)$ are of the form

$$\begin{aligned} & g_0^{-1}(g^{-1})^{(1)}\left((\partial g_0)^2 + \partial g_0 \partial u_0 \partial_\theta g^{(1)} + \left(\partial u_0 \partial_\theta g^{(1)}\right)^2\right) \\ & \quad + g_0^{-2}\partial g_0\left(\partial g^{(1)} + \partial u_0 \partial_\theta g^{(2)}\right) + g_0^{-2}\partial u_0 \partial_\theta g^{(1)}\left(\partial g^{(1)} + \partial u_0 \partial_\theta g^{(2)}\right). \end{aligned} \tag{3.3.36}$$

We decompose $P^{(1)}$ into a part depending on $g^{(2)}$ and a part depending only on the background and on $g^{(1)}$, hence (3.3.33). Next, we can decompose the two terms in (3.3.33) in frequencies following (3.3.36). For $P_{\alpha\beta}^{(1)}\left[g^{(\leq 1)}\right]$, this gives (3.3.34). Since the frequencies θ and 2θ are admissible at the λ level, a formal expression of $P^{(1,1)}$ and $P^{(1,2)}$ is enough:

$$P^{(1,k)} = \left(g_0^{-1}(g^{-1})^{(1)}\partial g_0 + g_0^{-2}\partial g^{(1)}\right)\left(\partial g_0 + \partial u_0 \partial_\theta g^{(1)}\right) \tag{3.3.37}$$

for $k \in [\![1, 2]\!]$. The frequency 3θ is forbidden at the λ level, therefore we need the exact expression of $P^{(1,3)}$. It comes from the term $g_0^{-1}(g^{-1})^{(1)}\partial u_0 \partial_\theta g^{(1)} \partial u_0 \partial_\theta g^{(1)}$ and is given by

$$P_{\alpha\beta}^{(1,3)} = \partial_{(\alpha} u_0 \hat{P}_{\beta)}^{(1,3)} \tag{3.3.38}$$

with

$$\hat{P}^{(1,3)}_{\beta} = -\frac{1}{8}\partial_\beta u_0 g_0^{\mu\rho}(F^{(1)})^{\nu\sigma}F^{(1)}_{\rho\sigma}F^{(1)}_{\mu\nu}. \tag{3.3.39}$$

Let us now look at the part of $P^{(1)}$ depending on $g^{(2)}$, denoted in (3.3.33) by $P^{(1)}_{\alpha\beta}\left[g^{(2)}\right]$. We first decompose it into terms coming from the interaction between $g^{(2)}$ and the background, i.e terms of the form $g_0^{-2}\partial g_0 \partial u_0 \partial_\theta g^{(2)}$, and terms coming from the interaction between $g^{(2)}$ and $g^{(1)}$, i.e terms of the form $g_0^{-2}(\partial u_0)^2 \partial_\theta g^{(1)}\partial_\theta g^{(2)}$. This gives (3.3.35) with

$$\hat{P}^{(1)}_{\beta}\left[g^{(2)}\right] = \cos\left(\frac{u_0}{\lambda}\right)\hat{P}^{(1,1)}_{\beta}\left[g^{(2)}\right] + \sin\left(\frac{2u_0}{\lambda}\right)\hat{P}^{(1,2)}_{\beta}\left[g^{(2)}\right] + \cos\left(\frac{3u_0}{\lambda}\right)\hat{P}^{(1,3)}_{\beta}\left[g^{(2)}\right]. \tag{3.3.40}$$

We don't need a precise expression for $\hat{P}^{(1,i)}_{\beta}\left[g^{(2)}\right]$ for $i = 1, 2, 3$ and only write down the following schematic expression

$$\hat{P}^{(1,i)}_{\beta}\left[g^{(2)}\right] = g_0^{-2}\left(\partial g_0 + \partial u_0 F^{(1)}\right)\left(\mathfrak{F} + F^{(2,k)}\right). \tag{3.3.41}$$

□

Since $P^{(\geq 2)}$ will be absorbed by the wave equation for the remainder $\mathfrak{h}$, we don't need to know its frequency decomposition. As all the terms coming from $P_{\alpha\beta}(g)(\partial g, \partial g)$ it is of the form $g^{-2}\partial g \partial g$.

3.4 The hierarchy of equations

In this section, we present the equations that the different terms in (3.3.1) solve. We divide them into two groups of equations, the background system presented in Section 3.4.2 and the reduced system presented in Section 3.4.3. Note that $g^{(3)}$ is defined in Section 3.4.5 but doesn't solve a differential equation. The background and reduced system are choosen so that (3.3.1) satisfy the Einstein vacuum equations, this will be proved in Section 3.7.

Before we present the equations, we define the following transport operator:

$$\mathscr{L}_0 := -2\mathbf{D}_{L_0} + \Box_{g_0} u_0 \tag{3.4.1}$$

where we recall that $\mathbf{D}$ is the covariant derivative associated to g_0. It acts on tensor fields of all type, including scalar functions.

3.4.1 A guide to the hierarchy

The exact expression of the equations satisfied by the different terms in (3.3.1) are quite complicated and in this section we break down the role these equations play in cancelling the different parts of the Ricci tensor. First note that the assumptions that $\mathrm{Pol}\left(F^{(1)}\right) = 0$ (which has been made in Section 3.3.2) implies that there is no λ^{-1} terms in the Ricci tensor of g. Therefore, we have the following decomposition

$$R_{\alpha\beta}(g) = R^{(0)}_{\alpha\beta} + \lambda R^{(1)}_{\alpha\beta} + \lambda^2 R^{(\geq 2)}_{\alpha\beta}.x$$

The λ^0 terms in $R_{\alpha\beta}(g)$ are given by

$$2R^{(0)}_{\alpha\beta} = -W^{(0)}_{\alpha\beta} + P^{(0)}_{\alpha\beta} + \left(H^\rho \partial_\rho g_{\alpha\beta} + g_{\rho(\alpha}\partial_{\beta)} H^\rho\right)^{(0)}, \tag{3.4.2}$$

where these different terms are given by Lemmas 3.3.1, 3.3.7 and (3.3.25). In order to obtain $R^{(0)}_{\alpha\beta} = 0$, we proceed as follows:

- the non-oscillating terms in (3.4.2) are absorbed by the background equation (3.1.7) for g_0,
- the terms in $\sin\left(\frac{u_0}{\lambda}\right)$ (which corresponds to an admissible frequency) are absorbed by a transport equation for $F^{(1)}$ (see (3.4.4) below), except for the terms with the non-tangential structure which are absorbed by the polarization tensor of $F^{(2,1)}$,
- the terms in $\cos\left(\frac{2u_0}{\lambda}\right)$ (which corresponds to a forbidden frequency and all have the non-tangential structure thanks to the weak polarized null condition) are absorbed by the polarization tensor of $F^{(2,2)}$.

The polarization conditions that $F^{(2,1)}$ and $F^{(2,2)}$ needs to solve are precisely given in Section 3.4.4.2 below.

The λ^1 terms in $R_{\alpha\beta}(g)$ are given by

$$2R^{(1)}_{\alpha\beta} = -W^{(1)}_{\alpha\beta} + P^{(1)}_{\alpha\beta} + \left(H^\rho \partial_\rho g_{\alpha\beta} + g_{\rho(\alpha}\partial_{\beta)} H^\rho\right)^{(1)}, \tag{3.4.3}$$

where these different terms are given by Lemmas 3.3.2, 3.3.8 and (3.3.26). In order to obtain $R^{(1)}_{\alpha\beta} = 0$, we proceed as follows. First we look at the terms independent from $\mathfrak{h}$ or $\mathfrak{F}$:

- the terms in $\cos\left(\frac{u_0}{\lambda}\right)$ and $\sin\left(\frac{2u_0}{\lambda}\right)$ (which corresponds to admissible frequencies) are absorbed by transport equations for $F^{(2,1)}$ and $F^{(2,2)}$ (see (3.4.5) and (3.4.6) below), except for the terms with a non-tangential structure which are absorbed by the polarization tensor of $g^{(3)}$,
- the terms in $\cos\left(\frac{3u_0}{\lambda}\right)$ (which corresponds to a forbidden frequency) with the non-tangential structure are absorbed by the polarization tensor of $g^{(3)}$,
- the term $\cos\left(\frac{3u_0}{\lambda}\right)$ without the non-tangential structure (see (3.3.10)) rewrites itself as a polarization term (see (3.4.17)) which eventually vanish at the end of the proof.

The polarization condition that $g^{(3)}$ needs to solve is precisely given in Section 3.4.5. The main terms depending on $\mathfrak{h}$ and $\mathfrak{F}$ are all multiplied by $\cos\left(\frac{u_0}{\lambda}\right)$ and are absorbed by the transport equation for $\mathfrak{F}$ (see (3.4.7) below).

In order to solve the Einstein vacuum equations, it remains to obtain $R^{(\geq 2)}_{\alpha\beta} = 0$. This is done by putting all the remaining terms on the RHS of a wave equation for $\mathfrak{h}$ (see (3.4.8) below) except for the Υ^ρ terms coming from the λ^2 terms in (3.3.17).

Remark 3.4.1. *Following this procedure, we don't actually solve the full Einstein vacuum equations. It will remain some polarization conditions and generalised wave gauge terms which are proved to vanish in Section 3.7.*

3.4.2 The background system

We present now the system solved by the background perturbations, called the *background system*. The equation for $F^{(1)}$ is

$$\mathscr{L}_0 F^{(1)}_{\alpha\beta} = 0. \tag{3.4.4}$$

The equations for $g^{(2)}$ are

$$\mathscr{L}_0 F^{(2,1)}_{\alpha\beta} = -\tilde{W}^{(1,1)}_{\alpha\beta} + P^{(1,1)}_{\alpha\beta} - \mathbf{D}_{(\alpha}\hat{P}^{(0,1)}_{\beta)}\left[g^{(1)}\right], \tag{3.4.5}$$

$$\mathscr{L}_0 F^{(2,2)}_{\alpha\beta} = -\tilde{W}^{(1,2)}_{\alpha\beta} - \frac{1}{2}P^{(1,2)}_{\alpha\beta} + \frac{1}{4}\left(F^{(2,1)}_{L_0L_0} + L_0^\sigma \hat{P}^{(0,1)}_\sigma\left[g^{(1)}\right]\right)F^{(1)}_{\alpha\beta} + \frac{1}{4}\mathbf{D}_{(\alpha}\hat{P}^{(0,2)}_{\beta)}\left[g^{(1)}\right] \tag{3.4.6}$$

The background system is a coupled transport-wave system which admits a triangular structure. It will be solved in Section 3.5 and all the estimates and support properties of the background perturbations are given in Theorem 3.5.1. The purpose of the equations (3.4.4)-(3.4.6) is given in Section 3.4.1. Their exact expression follows from the computations performed in Section 3.3.2.

Remark 3.4.2. *Note that the terms*

$$-\mathbf{D}_{(\alpha}\hat{P}^{(0,1)}_{\beta)}\left[g^{(1)}\right]$$

and

$$\frac{1}{4}L_0^\sigma \hat{P}^{(0,1)}_\sigma\left[g^{(1)}\right]F^{(1)}_{\alpha\beta} + \frac{1}{4}\mathbf{D}_{(\alpha}\hat{P}^{(0,2)}_{\beta)}\left[g^{(1)}\right]$$

in (3.4.5) and (3.4.6) are obtained from the gauge terms (3.3.26) after having replaced the polarization tensors $\mathrm{Pol}\left(F^{(2,i)}\right)$ *coming from* $H^{(1)}$ *by the non-tangential terms they absorbed in* $R^{(0)}_{\alpha\beta}$.

Since (3.4.4), (3.4.5) and (3.4.6) are first order transport equation, the initial data for the background system consist in

$$F^{(1)}_{\alpha\beta} \restriction \Sigma_0 \quad \text{and} \quad F^{(2,i)}_{\alpha\beta} \restriction \Sigma_0.$$

They are given in Section 3.4.6.1.

3.4.3 The reduced system

We present now the system solved by the tensors $\mathfrak{F}$ and $\mathfrak{h}$, called the *reduced system*. As explained in Section 3.2.2.3, we introduce the Fourier cut-offs $\Pi_\leq$ and $\Pi_\geq$:

Definition 3.4.1. *Let* $\chi_1 : \mathbb{R}^3 \longrightarrow [0,1]$ *be a smooth function supported in*

$$\left\{\xi \in \mathbb{R}^3 \;\middle|\; |\xi| \leq 2\right\}$$

and such that $\chi_1(\xi) = 1$ *if* $|\xi| \leq 1$. *For* $\lambda > 0$ *we define* $\chi_\lambda(\xi) = \chi_1(\lambda\xi)$, *thus* χ_λ *is supported in* $\left\{\xi \in \mathbb{R}^3 \;\middle|\; |\xi| \leq \frac{2}{\lambda}\right\}$ *and* $\chi_\lambda(\xi) = 1$ *if* $|\xi| \leq \frac{1}{\lambda}$. *We define* $\Pi_\leq$ *by*

$$\Pi_\leq(f) = \mathcal{F}^{-1}\left(\chi_\lambda \mathcal{F}(f)\right)$$

for $f \in L^2$. *We also define*

$$\Pi_\geq = \mathrm{Id} - \Pi_\leq.$$

The equations for $\mathfrak{F}$ and $\mathfrak{h}$ are then

$$\mathscr{L}_0\mathfrak{F}_{\alpha\beta} = -\Pi_{\leq}\left(\mathfrak{h}_{L_0L_0}\right)F^{(1)}_{\alpha\beta}, \tag{3.4.7}$$

$$\tilde{\Box}_g\mathfrak{h}_{\alpha\beta} = -\sin\left(\frac{u_0}{\lambda}\right)\tilde{\Box}_g\mathfrak{F}_{\alpha\beta} - \lambda\tilde{\Box}_g g^{(3)}_{\alpha\beta} - \frac{1}{\lambda}\cos\left(\frac{u_0}{\lambda}\right)\Pi_{\geq}\left(\mathfrak{h}_{L_0L_0}\right)F^{(1)}_{\alpha\beta} + \mathcal{R}(g) \tag{3.4.8}$$

where we set

$$\mathcal{R}(g) = -\tilde{W}^{(\geq 2)}_{\alpha\beta} + \left(\mathring{H}^\rho\partial_\rho g_{\alpha\beta} + g_{\rho(\alpha}\partial_{\beta)}\mathring{H}^\rho\right)^{(\geq 2)} + P^{(\geq 2)}_{\alpha\beta}. \tag{3.4.9}$$

with $\mathring{H}$ defined in (3.3.18).

Equations (3.4.7)-(3.4.8) form a coupled transport-wave system and its purpose is to solve the tangential part of the remaining orders of the Ricci tensor of g, i.e some part of its order 1 and the orders superior or equal to 2, as explained in Section 3.4.1. The spectral projectors $\Pi_{\leq}$ and $\Pi_{\geq}$ are used to avoid the loss of derivatives described in the introduction, see Section 3.2.2.3.

Since (3.4.7) is a first order transport equation and (3.4.8) is a second order wave equation the initial data for the reduced system consist in

$$\mathfrak{F}_{\alpha\beta}\restriction\Sigma_0, \quad \mathfrak{h}_{\alpha\beta}\restriction\Sigma_0 \quad \text{and} \quad \partial_t\mathfrak{h}_{\alpha\beta}\restriction\Sigma_0.$$

They are given in Section 3.4.6.2.

3.4.4 Generalised wave gauge and polarization conditions

As it is usual when solving the Einstein equations, we impose gauge conditions. Since we solve a wave equation for $\mathfrak{h}$ we need a generalised wave gauge. Moreover, as the study of first order high-frequency waves by Choquet-Bruhat shows, the oscillating parts of the ansatz need to satisfy polarization conditions, i.e their polarization tensor (see (3.2.8) for its definition) is prescribed.

3.4.4.1 Generalised wave gauge condition

The tensor $\mathfrak{h}$ solves the wave equation (3.4.8) and we associate to it a generalised wave gauge condition. In Section 3.3.2.2 we already defined Υ^ρ, see (3.3.16).

Definition 3.4.2. *The generalised wave gauge considered in Theorem 3.1.2 is defined by*

$$\Upsilon^\rho = 0. \tag{3.4.10}$$

Remark 3.4.3. *For later use, we define $\tilde{\Upsilon}^\rho$ as the terms in Υ^ρ independent from $\partial\mathfrak{h}$:*

$$\begin{aligned}\tilde{\Upsilon}^\rho = g^{\mu\nu}g^{\rho\sigma}&\left(\sin\left(\frac{u_0}{\lambda}\right)\left(\partial_\mu\mathfrak{F}_{\sigma\nu} - \frac{1}{2}\partial_\sigma\mathfrak{F}_{\mu\nu}\right) + \lambda\left(\partial_\mu g^{(3)}_{\sigma\nu} - \frac{1}{2}\partial_\sigma g^{(3)}_{\mu\nu}\right)\right) \\ &- g_0^{\rho\sigma}\mathfrak{h}^{\mu\nu}\left(\partial_\mu(g_0)_{\sigma\nu} - \frac{1}{2}\partial_\sigma(g_0)_{\mu\nu} - \sin\left(\frac{u_0}{\lambda}\right)\left(\partial_\mu u_0F^{(1)}_{\sigma\nu} - \frac{1}{2}\partial_\sigma u_0F^{(1)}_{\mu\nu}\right)\right).\end{aligned} \tag{3.4.11}$$

3.4.4.2 Polarization conditions

The polarization conditions prescribe the polarization tensor of $F^{(2,i)}$ in terms of $F^{(1)}$ and the background metric.

Definition 3.4.3. *We define the following tensors:*

$$V_\beta^{(2,1)} = \mathrm{Pol}_\beta\left(F^{(2,1)}\right) + (g_0)_{\rho\beta}(\tilde{H}^{(1,1)})^\rho + \hat{P}_\beta^{(0,1)}\left[g^{(1)}\right], \tag{3.4.12}$$

$$V_\beta^{(2,2)} = \mathrm{Pol}_\beta\left(F^{(2,2)}\right) - \frac{1}{2}(g_0)_{\rho\beta}(\tilde{H}^{(1,2)})^\rho - \frac{1}{4}\hat{P}_\beta^{(0,2)}\left[g^{(1)}\right]. \tag{3.4.13}$$

The polarization conditions are

$$V_\beta^{(2,j)} = 0 \tag{3.4.14}$$

together with

$$\mathrm{Pol}_\beta\left(F^{(1)}\right) = 0. \tag{3.4.15}$$

We don't introduce the notation $V^{(1)}$ since the proof of (3.4.15) and (3.4.14) are different. They are both based on the same idea: computing transport equations satisfied by either $\mathrm{Pol}_\beta\left(F^{(1)}\right)$ or $V^{(2,j)}$ and use the fact that those quantities vanish on Σ_0 (see Corollary 3.4.1 below). However, the method used to derive those transport equations differs, see Section 3.2.1.4 for more details.

The exact expression of $V^{(2,1)}$ is irrelevant. However, (3.3.21) and (3.3.32) imply

$$L_0^\beta\left(-\frac{1}{2}(g_0)_{\rho\beta}(\tilde{H}^{(1,2)})^\rho - \frac{1}{4}\hat{P}_\beta^{(0,2)}\left[g^{(1)}\right]\right) = 0 \tag{3.4.16}$$

which together with (3.2.9) and (3.4.13) implies

$$F_{L_0L_0}^{(2,2)} = -V_{L_0}^{(2,2)}. \tag{3.4.17}$$

This shows how we deal with the forbidden frequency coming from the quasi-linearity in the wave part of the Ricci tensor. The frequency 3θ at the λ level (see (3.3.10)) is proportional to $F_{L_0L_0}^{(2,2)}$ which thanks to (3.4.17) rewrite as V terms, i.e it vanishes if we are able to prove that the polarization conditions (3.4.14) hold.

3.4.5 Definition of $g^{(3)}$

The tensor $g^{(3)}$ is introduced in the ansatz (3.3.1) in order to absorb the non-tangential part of the λ^1 level in the Ricci tensor of g. Unlike the other terms in (3.3.1) it doesn't solve any differential equation but only a polarization condition. More precisely we want $g^{(3)}$ to satisfy

$$\begin{aligned}\mathrm{Pol}_\beta\left(\partial_\theta^2 g^{(3)}\right) &= -(g_0)_{\rho\beta}\partial_\theta(\tilde{H}^{(2)})^\rho - \cos\left(\frac{u_0}{\lambda}\right)F_{\rho\beta}^{(1)}\partial_\theta(H^{(1)})^\rho \\ &\quad - \hat{P}_\beta^{(1)}\left[g^{(2)}\right] - \cos\left(\frac{3u_0}{\lambda}\right)\hat{P}_\beta^{(1,3)}\end{aligned} \tag{3.4.18}$$

A necessary and sufficient condition for (3.4.18) to be satisfied is that its RHS is purely oscillating, this holds thanks to (3.3.19) and (3.3.40). Indeed, (3.3.40) implies that $\hat{P}_\beta^{(1)}\left[g^{(2)}\right]$

is purely oscillating while (3.3.19) implies that the product $\cos\left(\frac{u_0}{\lambda}\right)\partial_\theta(H^{(1)})^\rho$ is also purely oscillating.

We are left with the question of solving an equation of the form

$$\mathrm{Pol}_\beta(T) = \Omega_\beta \tag{3.4.19}$$

with Ω_β a 1-form for T a symmetric 2-tensor. According to (3.2.9), (3.2.10) and (3.2.11) we just need to set

$$\begin{aligned} T_{L_0L_0} &= -\Omega_{L_0} \\ T_{L_0A} &= -\Omega_A \\ T_{11} &= -\Omega_{\underline{L}_0} \end{aligned}$$

with all the other coefficients of T being zero. This implies that we can solve (3.4.19) for T satisfying $|T| \lesssim |\Omega|$ with an implicit constant depending only on the background spacetime.

As it is clear from the RHS of (3.4.18), $g^{(3)}$ depends on the background quantities g_0, $F^{(1)}$, $F^{(2,i)}$ and their first derivatives and on the non-background field $\mathfrak{F}$ through the terms $\tilde{H}^{(2)}$ and $\hat{P}^{(1)}_\beta\left[g^{(2)}\right]$ (see (3.3.23) and (3.3.41)). However, it does not depend on $\mathfrak{h}$. We summarize this discussion in the next lemma.

Lemma 3.4.1. *Given $F^{(1)}$, $F^{(2,i)}$ and $\mathfrak{F}$, there exists a symmetric 2-tensor $g^{(3)}$ solution of (3.4.18) of the form*

$$g^{(3)} = \sum_{\mathrm{T}\in\mathcal{A}} \mathrm{T}\left(\frac{u_0}{\lambda}\right)\left(g^{(3,\mathrm{T})}(\mathfrak{F}) + g^{(3,\mathrm{T},\mathrm{BG})}\right)$$

where $\mathcal{A}$ is a finite subset of trigonometric functions (see (3.2.4)), $g^{(3,\mathrm{T})}(\mathfrak{F})$ satisfy schematically

$$g^{(3,\mathrm{T})}(\mathfrak{F}) = g_0 g_0^{-3}\left(\partial g_0 + \partial u_0 F^{(1)}\right)\mathfrak{F} \tag{3.4.20}$$

and where $g^{(3,\mathrm{T},\mathrm{BG})}$ is a polynomial in the background quantities g_0, $F^{(1)}$, $F^{(2,i)}$ and their first derivatives. In the rest of this article, we will use the notation

$$\begin{aligned} g^{(3)}(\mathfrak{F}) &= \sum_{\mathrm{T}\in\mathcal{A}} \mathrm{T}\left(\frac{u_0}{\lambda}\right) g^{(3,\mathrm{T})}(\mathfrak{F}), \\ g^{(3,\mathrm{BG})} &= \sum_{\mathrm{T}\in\mathcal{A}} \mathrm{T}\left(\frac{u_0}{\lambda}\right) g^{(3,\mathrm{T},\mathrm{BG})}. \end{aligned}$$

Note that since $g^{(3)}$ only absorb oscillating terms, it has the same support as $F^{(1)}$, $F^{(2,i)}$ or $\mathfrak{F}$, which will be proved to be compact in this chapter.

3.4.6 Initial data for the spacetime metric

In this section, we define the initial data on Σ_0 for the different tensors appearing in the high-frequency ansatz (3.3.1). They are based on the solution of the constraint equations presented in Theorem 3.1.1, which is the main result of Chapter 2.

3.4.6.1 Initial data for the background system

We start with the tensors $F^{(1)}$, $F^{(2,1)}$ and $F^{(2,2)}$. For the spatial components, we simply set

$$F_{ij}^{(1)} = \bar{F}_{ij}^{(1)}, \quad F_{ij}^{(2,1)} = \bar{F}_{ij}^{(2,1)}, \quad F_{ij}^{(2,2)} = \bar{F}_{ij}^{(2,2)}. \tag{3.4.21}$$

For the time components of $F^{(1)}$, we set

$$F_{0\alpha}^{(1)} \restriction \Sigma_0 = 0. \tag{3.4.22}$$

For the time components of $F^{(2,i)}$, we need to ensure that $F^{(2,i)}$ satisfy on Σ_0 the polarization conditions (3.4.14). For clarity, we define the 1-forms $\Omega^{(i)}$ for $i = 1, 2$ so that

$$V_\beta^{(2,i)} = \mathrm{Pol}_\beta\left(F^{(2,i)}\right) - \Omega_\beta^{(i)}. \tag{3.4.23}$$

Note from (3.4.12), (3.4.13), (3.3.20), (3.3.21), (3.3.31) and (3.3.32) that $\Omega^{(i)}$ involve only g_0 and its derivatives or $F^{(1)}$ and its first derivatives $\partial F^{(1)}$. For $\nabla F^{(1)}$, the initial data on Σ_0 are obtained by differenciating directly $F_{\alpha\beta}^{(1)}$ defined in (3.4.21) and (3.4.22) and for $\partial_t F_{\alpha\beta}^{(1)}$ they are obtained by rewriting the equation we want $F^{(1)}$ to satisfy, that is (3.4.4) which rewrites as

$$\partial_t F_{\alpha\beta}^{(1)} = -N_0 F_{\alpha\beta}^{(1)} + (\partial_t + N_0)^\rho \Gamma(g_0)_{\rho(\alpha}^\mu F_{\mu\beta)}^{(1)} + \frac{1}{2|\nabla u_0|_{\bar{g}_0}} (\Box_{g_0} u_0) F_{\alpha\beta}^{(1)} \tag{3.4.24}$$

where we used the fact that on Σ_0 we have $L_0 = |\nabla u_0|_{\bar{g}_0} (\partial_t + N_0)$. At this stage, the RHS of (3.4.24) is fully defined on Σ_0, which implies that this is also the case for $\Omega^{(i)}$. It remains to define the initial data for $F_{0\alpha}^{(2,i)}$ for $i = 1, 2$ by

$$F_{0A}^{(2,i)} = -\bar{F}_{AN_0}^{(2,i)} - \frac{1}{|\nabla u_0|_{\bar{g}_0}} \Omega_A^{(i)}, \tag{3.4.25}$$

$$F_{0N_0}^{(2,i)} = -\frac{1}{2} \bar{F}_{N_0N_0}^{(2,i)} - \frac{1}{2|\nabla u_0|_{\bar{g}_0}^2} \Omega_{L_0}^{(i)}, \tag{3.4.26}$$

$$F_{00}^{(2,i)} = 0. \tag{3.4.27}$$

Note that these definitions only ensure

$$V_{\mathcal{T}_0}^{(2,i)} \restriction \Sigma_0 = 0 \tag{3.4.28}$$

thanks to (3.2.9) and (3.2.10). The fact that $V_{\underline{L}_0}^{(2,i)}$ also vanishes on Σ_0 is the content of Lemma 3.7.9.

3.4.6.2 Initial data for the reduced system

We define here the initial data for the solution of the reduced system. For the tensor $\mathfrak{F}$ we simply set

$$\mathfrak{F}_{\alpha\beta} \restriction \Sigma_0 = 0. \tag{3.4.29}$$

Before we look at $\mathfrak{h}$, note that with (3.4.29) we can apply Lemma 3.4.1 and fully define $g^{(3)}$ on Σ_0. Indeed it is independent from $\mathfrak{h}$ and depends only on $F^{(1)}$, $F^{(2,i)}$ and their derivatives which can be computed on Σ_0 by looking at the transport equations we want them to satisfy, that is (3.4.4), (3.4.5) and (3.4.6).

Since $\mathfrak{h}$ solves a second order equation we need to prescribe $\mathfrak{h}$ and $\partial_t \mathfrak{h}$ on Σ_0. We start with the spatial components of $\mathfrak{h}$, defined so that the induced metric on Σ_0 of the spacetime metric g defined by (3.3.1) is precisely the Riemannian metric $\bar{g}$ obtained in Theorem 3.1.1. This imposes

$$\mathfrak{h}_{ij} = \bar{\mathfrak{h}}_{ij} - \lambda g^{(3)}_{ij}. \tag{3.4.30}$$

where $\bar{h}$ is given in (3.1.16). For the time components of $\mathfrak{h}$, we choose them so that they compensate the one of $g^{(3)}$, that is

$$\mathfrak{h}_{0\alpha} = -\lambda g^{(3)}_{0\alpha}. \tag{3.4.31}$$

Note that we can now obtain the initial data for the derivatives of $\mathfrak{F}$. Indeed, (3.4.29) obviously gives us that $\nabla \mathfrak{F}_{\alpha\beta} \restriction \Sigma_0 = 0$ and since we want $\mathfrak{F}$ to solve (3.4.7), the time derivative of $\mathfrak{F}$ is initially given by

$$\partial_t \mathfrak{F}_{\alpha\beta} = \frac{1}{2|\nabla u_0|_{\bar{g}_0}} \Pi_{\leq} \left(\mathfrak{h}_{L_0 L_0}\right) F^{(1)}_{\alpha\beta} \tag{3.4.32}$$

where we also used (3.4.29) to simplify (3.4.7).

We now want to define the initial data for $\partial_t \mathfrak{h}_{\alpha\beta}$. The spatial components, i.e $\partial_t \mathfrak{h}_{ij}$, are chosen so that the second fundamental form of Σ_0 as an hypersurface in $(\mathcal{M}, g)$ is given by K obtained in Theorem 3.1.1. The second fundamental form is defined by $-\frac{1}{2}\mathcal{L}_T g$ where T is the unit normal to Σ_0 for g and $\mathcal{L}$ is the Lie derivative. *A priori*, the unit normal T depends on λ. This shows that in order to obtain the dependence of $\partial_t \mathfrak{h}_{ij}$ on λ, we need to know more about T.

Lemma 3.4.2. *The future unit normal T to Σ_0 for the metric g given by (3.3.1) satisfies*

$$T = \partial_t + Z \tag{3.4.33}$$

with the vector field $Z = Z^t \partial_t + \bar{Z}^i \partial_i$ given by

$$Z^t = \left(1 + \lambda^4 \bar{g}^{ij} g^{(2)}_{0i} g^{(2)}_{0j}\right)^{-\frac{1}{2}} - 1, \tag{3.4.34}$$

$$\bar{Z}^i = -\lambda^2 \bar{g}^{ij} g^{(2)}_{0j} \left(1 + \lambda^4 \bar{g}^{ij} g^{(2)}_{0i} g^{(2)}_{0j}\right)^{-\frac{1}{2}}. \tag{3.4.35}$$

Proof. Since the unit normal to a spacelike hypersurface for a given Lorentzian metric is unique, we just need to prove that we can construct the vector field Z such that T defined by (3.4.33) satisfies

$$g(T,T) = -1 \quad \text{and} \quad g(T, \partial_i) = 0.$$

Note that thanks to (3.4.22), (3.4.27), (3.4.29) and (3.4.31) and the fact that ∂_t is the unit normal to Σ_0 for g_0 we have $g_{00} = -1$ and $g_{0i} = \lambda^2 g^{(2)}_{0i}$. This gives

$$g(T, \partial_i) = \lambda^2 g^{(2)}_{0i} \left(1 + Z^t\right) + \bar{g}_{ij} \bar{Z}^j$$

so we impose

$$\bar{Z}^i = -\lambda^2 \bar{g}^{ij} g^{(2)}_{0j} \left(1 + Z^t\right). \tag{3.4.36}$$

Moreover, we have

$$g(T,T) = -1 + 2g_{0\alpha}Z^\alpha + g_{\alpha\beta}Z^\alpha Z^\beta$$

so we impose $2g_{0\alpha}Z^\alpha + g_{\alpha\beta}Z^\alpha Z^\beta = 0$. After expanding the quadratic term and inserting (3.4.36) this condition rewrites as a quadratic equation for Z^t

$$\left(1+\lambda^4\bar{g}^{ij}g_{0i}^{(2)}g_{0j}^{(2)}\right)\left(Z^t\right)^2 + 2\left(1+\lambda^4\bar{g}^{ij}g_{0i}^{(2)}g_{0j}^{(2)}\right)Z^t + \lambda^4\bar{g}^{ij}g_{0i}^{(2)}g_{0j}^{(2)} = 0.$$

The two roots of this equation are

$$\pm\left(1+\lambda^4\bar{g}^{ij}g_{0i}^{(2)}g_{0j}^{(2)}\right)^{-\frac{1}{2}} - 1.$$

Since we want T to be future directed, we choose the root where $\pm = +$, i.e (3.4.34). Inserting this into (3.4.36) gives (3.4.35). □

We now compute the expression of the second fundamental form of Σ_0 as an hypersurface in $(\mathcal{M}, g)$ with g being given by (3.3.1).

Lemma 3.4.3. *If g is given by (3.3.1) and if $\partial_t F_{\alpha\beta}^{(1)}$ is given by (3.4.24) then on Σ_0 the following holds*

$$-\frac{1}{2}\mathcal{L}_T g_{ij} = K_\lambda^{(0)} + \lambda K_\lambda^{(1)} + \lambda^2\left(-\frac{1}{2}\mathcal{L}_T\mathfrak{h} + \tilde{K}_{evol}^{(\geq 2)}\right)$$

where $K_\lambda^{(0)}$ and $K_\lambda^{(1)}$ are given by (3.1.18) and (3.1.19). Moreover, we have schematically

$$\lambda^2\tilde{K}_{evol}^{(\geq 2)} = Z^\alpha\partial g_0 + \lambda Z^\alpha\partial g^{(1)} + \lambda^2\left(\partial\mathfrak{F} + \partial F^{(2,i)} + Z^\alpha\partial g^{(2)}\right) + \lambda^3 T g^{(3)}. \tag{3.4.37}$$

Proof. Recalling that g is defined by (3.3.1) we have

$$-\frac{1}{2}\mathcal{L}_T g = -\frac{1}{2}\mathcal{L}_T g_0 - \frac{\lambda}{2}\mathcal{L}_T g^{(1)} - \frac{\lambda^2}{2}\mathcal{L}_T g^{(2)} - \frac{\lambda^2}{2}\mathcal{L}_T\mathfrak{h} - \frac{\lambda^3}{2}\mathcal{L}_T g^{(3)}.$$

We compute or schematically estimate each term in this expression, using the expansion of T given in Lemma 3.4.2. The latter gives

$$-\frac{1}{2}\mathcal{L}_T g_0 = K_0 + O\left(Z^\alpha\partial g_0\right)$$

where Z^α denotes either Z^t or $\bar{Z}^i$ and where we used the fact that ∂_t is the unit normal to Σ_0 for g_0. Similarly, using $\partial_t u_0 = |\nabla u_0|_{\bar{g}_0}$ on Σ_0 we have

$$\begin{aligned}
-\frac{\lambda}{2}\mathcal{L}_T g_{ij}^{(1)} &= \frac{1}{2}\sin\left(\frac{u_0}{\lambda}\right)|\nabla u_0|_{\bar{g}_0}\bar{F}_{ij}^{(1)} - \frac{\lambda}{2}\cos\left(\frac{u_0}{\lambda}\right)\partial_t F_{ij}^{(1)} + O\left(\lambda Z^\alpha\partial g^{(1)}\right)\\
&= \frac{1}{2}\sin\left(\frac{u_0}{\lambda}\right)|\nabla u_0|_{\bar{g}_0}\bar{F}_{ij}^{(1)}\\
&\quad - \frac{\lambda}{2}\cos\left(\frac{u_0}{\lambda}\right)\left(-N_0\bar{F}_{ij}^{(1)} + (\partial_t + N_0)^\rho\Gamma(g_0)_{\rho(i}^k\bar{F}_{kj)}^{(1)} + \frac{1}{2|\nabla u_0|_{\bar{g}_0}}(\Box_{g_0}u_0)\bar{F}_{ij}^{(1)}\right)\\
&\quad + O\left(\lambda Z^\alpha\partial g^{(1)}\right)
\end{aligned}$$

where we used (3.4.24) and (3.4.22) to compute $\partial_t F^{(1)}_{ij}$ and (3.4.21) to compute $F^{(1)}_{ij}$. Using in addition (3.4.29) we now obtain

$$\begin{aligned}-\frac{\lambda^2}{2}\mathcal{L}_T g^{(2)}_{ij} &= -\frac{\lambda}{2}\cos\left(\frac{u_0}{\lambda}\right)|\nabla u_0|_{\bar{g}_0}\bar{F}^{(2,1)}_{ij} + \lambda\sin\left(\frac{2u_0}{\lambda}\right)|\nabla u_0|_{\bar{g}_0}\bar{F}^{(2,2)}_{ij} \\ &\quad + O\left(\lambda^2\left(\partial\mathfrak{F} + \partial F^{(2,i)} + Z^\alpha \partial g^{(2)}\right)\right).\end{aligned}$$

This concludes the proof of the lemma, by recalling the expression of $K^{(i)}_\lambda$ for $i = 1, 2$ given in (3.1.18) and (3.1.19). □

As explained above, we want the initial data for $\partial_t \mathfrak{h}_{ij}$ to ensure that the second fundamental form of Σ_0 for g, i.e $-\frac{1}{2}\mathcal{L}_T g_{ij}$, matches the tensor K_λ given by Theorem 3.1.1. According to the previous lemma, this is equivalent to

$$-\frac{1}{2}\mathcal{L}_T\mathfrak{h} + \tilde{K}^{(\geq 2)}_{evol} = K^{(\geq 2)}_\lambda \tag{3.4.38}$$

where $K^{(\geq 2)}_\lambda$ is given in Theorem 3.1.1. Using the expansion of T, this defines the initial data for $\partial_t \mathfrak{h}_{ij}$ on Σ_0 by

$$\partial_t \mathfrak{h}_{ij} = \frac{1}{1+Z^t}\left(-\bar{Z}^k\partial_k\mathfrak{h}_{ij} + \mathfrak{h}\left([Z,\partial_{(i}],\partial_{j)}\right) - 2(K^{(\geq 2)}_\lambda)_{ij} + 2(\tilde{K}^{(\geq 2)}_{evol})_{ij}\right). \tag{3.4.39}$$

It remains to define the initial value for $\partial_t \mathfrak{h}_{0\alpha}$. They are chosen so that the generalised wave condition (3.4.10) holds on Σ_0. This condition rewrites as

$$g^{\mu\nu}\left(\partial_\mu \mathfrak{h}_{\sigma\nu} - \frac{1}{2}\partial_\sigma \mathfrak{h}_{\mu\nu}\right) = -g_{\sigma\rho}\tilde{\Upsilon}^\rho. \tag{3.4.40}$$

Note that the initial data for $\tilde{\Upsilon}^\rho$ are fully defined. Indeed the last line in (3.4.11) depends on background quantities and on $g \restriction \Sigma_0$. For the first line, we use (3.4.29) for the tangential derivatives or (3.4.32) for the initial value of $\partial_t\mathfrak{F}$. A similar argument using Lemma 3.4.1 allows us to deal with $\partial g^{(3)}$ in $\tilde{\Upsilon}^\rho$.

Thanks to the decomposition of $g^{\mu\nu}$ on Σ_0 (3.4.40) rewrites as

$$-T^\mu T^\nu\left(\partial_\mu \mathfrak{h}_{\sigma\nu} - \frac{1}{2}\partial_\sigma \mathfrak{h}_{\mu\nu}\right) = -g_{\sigma\rho}\tilde{\Upsilon}^\rho - \bar{g}^{k\ell}\left(\partial_k \mathfrak{h}_{\sigma\ell} - \frac{1}{2}\partial_\sigma \mathfrak{h}_{k\ell}\right).$$

where the RHS is fully defined thanks to (3.4.30), (3.4.31) and (3.4.39). We project this expression onto the initial frame (T, ∂_i) and rearrange terms to obtain the initial value of $T\mathfrak{h}_{TT}$ and $T\mathfrak{h}_{iT}$:

$$T\mathfrak{h}_{TT} = 2\mathfrak{h}_{\mu T} T T^\mu + 2g_{T\rho}\tilde{\Upsilon}^\rho + 2T^\sigma \bar{g}^{k\ell}\left(\partial_k \mathfrak{h}_{\sigma\ell} - \frac{1}{2}\partial_\sigma \mathfrak{h}_{k\ell}\right), \tag{3.4.41}$$

$$T\mathfrak{h}_{iT} = \mathfrak{h}_{i\nu} T T^\nu + \frac{1}{2}T^\mu T^\nu \partial_i \mathfrak{h}_{\mu\nu} + g_{i\rho}\tilde{\Upsilon}^\rho + \bar{g}^{k\ell}\left(\partial_k \mathfrak{h}_{i\ell} - \frac{1}{2}\partial_i \mathfrak{h}_{k\ell}\right). \tag{3.4.42}$$

Using the decomposition of T given by Lemma 3.4.2 we can deduce from (3.4.41) and (3.4.42) the initial value of $\partial_t \mathfrak{h}_{0\alpha}$ (see how we obtain (3.4.39) from (3.4.38)).

3.4.6.3 Initial estimates and properties

In the following corollary, we summarize the formal properties satisfied by the initial data defined in the previous sections and give the estimates they satisfy.

Corollary 3.4.1. *The initial data for the background and reduced systems defined in Sections 3.4.6.1 and 3.4.6.2 are such that*

(i) the first and second fundamental forms of the hypersurface Σ_0 are given by $(\bar{g}, K)$ from Theorem 3.1.1 and therefore solve the constraint equations,

(ii) the tensors $F^{(1)}$, $F^{(2,1)}$ and $F^{(2,2)}$ are initially supported in $\{|x| \leq R\}$ and there exists a constant $C_{\mathrm{in}} > 0$ such that

$$\left\| F^{(1)} \right\|_{H^N(\Sigma_0)} + \left\| F^{(2,1)} \right\|_{H^{N-1}(\Sigma_0)} + \left\| F^{(2,2)} \right\|_{H^{N-1}(\Sigma_0)} \leq C_{\mathrm{in}} \varepsilon, \tag{3.4.43}$$

$$\max_{k \in [\![0,4]\!]} \lambda^r \left\| \partial \nabla^r \mathfrak{h} \right\|_{L^2_{\delta+r+1}(\Sigma_0)} \leq C_{\mathrm{in}} \varepsilon, \tag{3.4.44}$$

(iii) the following initial polarization conditions hold

$$\mathrm{Pol}\left(F^{(1)} \right) = 0, \tag{3.4.45}$$

$$V^{(2,i)}_{T_0} = 0, \tag{3.4.46}$$

$$\mathrm{Pol}\left(\mathfrak{F} \right) = 0, \tag{3.4.47}$$

(iv) the initial backreaction condition hold

$$\mathcal{E}\left(F^{(1)}, F^{(1)} \right) = -4 F_0^2, \tag{3.4.48}$$

where $\mathcal{E}$ is defined in (3.2.16),

(v) the initial generalised wave gauge condition hold

$$\Upsilon^\rho = 0. \tag{3.4.49}$$

Proof. The first item of the corollary holds thanks to (3.4.21), (3.4.29), (3.4.30) and (3.4.38). We now turn to the second item the corollary. The support property of $F^{(1)}$, $F^{(2,1)}$ and $F^{(2,2)}$ follows directly from the support property of $\bar{F}^{(1)}$, $\bar{F}^{(2,1)}$ and $\bar{F}^{(2,2)}$ stated in Theorem 3.1.1 and (3.4.21), (3.4.22), (3.4.25), (3.4.26) and (3.4.27). Similarly, we deduce (3.4.43) from (3.1.20).

Proving (3.4.44) amounts to bound the Sobolev norm of the initial data for $\mathfrak{h}$ and $\partial_t \mathfrak{h}$. First, from (3.4.30) and (3.4.31) we can schematically write $\mathfrak{h} = \bar{\mathfrak{h}} + \lambda g^{(3)}$. On Σ_0, $g^{(3)}$ is compactly supported and depends only on background quantities (see the discussion following (3.4.29)). The estimate

$$\left\| \nabla^{r+1} \mathfrak{h} \right\|_{L^2_{\delta+r+1}(\Sigma_0)} \lesssim \frac{\varepsilon}{\lambda^r} \tag{3.4.50}$$

for $r \in [\![0,4]\!]$ thus follows from (3.1.22). We now look at $\partial_t \mathfrak{h}_{ij}$, defined in (3.4.39). Since $Z = O\left(\lambda^2\right)$ and only contains background quantities (see (3.4.35) and (3.4.34)) the worse terms in (3.4.39) are the last two (the first two can be estimated with (3.4.50)). For them we use (3.1.23) and (3.4.37) (note that $\tilde{K}^{(\geq 2)}_{evol}$ involves the non-background quantities $\partial_t \mathfrak{F}$ for which we need to use (3.4.32)). This proves that

$$\|\partial_t \nabla^r \mathfrak{h}_{ij}\|_{L^2_{\delta+r+1}(\Sigma_0)} \lesssim \frac{\varepsilon}{\lambda^r}$$

for $r \in [\![0,4]\!]$. We proceed similarly for $\partial_t \mathfrak{h}_{0\alpha}$ using (3.4.41) and (3.4.42). This concludes the proof of (3.4.44).

The initial polarization conditions (3.4.46) and (3.4.47) are consequences of (3.4.25)-(3.4.26)-(3.4.27) and (3.4.29), while (3.4.45) holds since $F^{(1)}$ is initially $P_{0,u}$-tangent and g_0-traceless (see the corresponding properties of $\bar{F}^{(1)}$ in Theorem 3.1.1). Samewise, (3.4.48) follows from (3.1.21). The initial generalised wave gauge condition (3.4.49) follows from (3.4.41) and (3.4.42). □

3.5 Solving the background system

In this section we solve the background system (3.4.4)-(3.4.6) defined in Section 3.4.2. The following theorem summarizes the estimates and the support properties of the background perturbations.

Theorem 3.5.1. *Given initial data as in Corollary 3.4.1 and if ε is small enough, there exists a unique solution*

$$\left(F^{(1)}, F^{(2,1)}, F^{(2,2)}\right) \tag{3.5.1}$$

of the background system (3.4.4)-(3.4.6) on $[0,1]\times\mathbb{R}^3$. Moreover:

(i) the tensors in (3.5.1) are supported in $\{|x| \leq C_{\text{supp}} R\}$,

(ii) there exists $C_1 = C_1(C_0) > 0$ such that the following estimates hold

$$\left\|F^{(1)}\right\|_{H^N} + \left\|F^{(2,1)}\right\|_{H^{N-2}} + \left\|F^{(2,2)}\right\|_{H^{N-2}} \leq C_1\varepsilon,$$

(iii) $F^{(1)}$ satisfies (3.1.26) and (3.1.27).

The proof of this theorem is the content of Sections 3.5.2 and 3.5.3.

3.5.1 Background transport estimates

We want the background perturbations to solve the equations (3.4.4)-(3.4.5)-(3.4.6). The equations solved by their coefficients in any basis are thus of the form

$$(L_0 + \eta)\, h = f \tag{3.5.2}$$

for f and η given scalar functions on $[0,1]\times\mathbb{R}^3$ (recall (3.4.1)). Moreover, thanks to Corollary 3.4.1 the initial data on Σ_0 for the background perturbations are compactly supported. For the model equation (3.5.2) this allows us to consider f supported in $[0,1]\times K$ for K a compact of $\mathbb{R}^3$ and $h \restriction \Sigma_0$ compactly supported. Since L_0 is the spacetime gradient of the background phase u_0 and thanks to (3.1.3) and (3.1.9) we can define characteristics for the equation (3.5.2) on $[0,1]\times\mathbb{R}^3$. The existence of a solution h to this equation can thus be proved using the method of characteristics, we don't give the details.

A solution h of (3.5.2) can be proved to be as regular as f using its expression given by the characteristics method. However, since our proof of Theorem 3.1.2 is based on Sobolev spaces, we now derive energy estimates for the transport operator L_0. As a byproduct, this standard estimate proves the unicity of a solution to (3.5.2) and therefore to the solutions of (3.4.4)-(3.4.5)-(3.4.6).

Lemma 3.5.1. *Let h and f two scalar functions on $[0,T]\times\mathbb{R}^3$ for some $T\leq 1$, compactly supported and such that*

$$L_0 h = f. \tag{3.5.3}$$

Then the following holds for all $t\in[0,T]$

$$\|h\|^2_{L^2}(t) \leq C(C_0)\left(\|h\|^2_{L^2}(0) + \int_0^t \left(\|h\|^2_{L^2}(s) + \|f\|^2_{L^2}(s)\right)\mathrm{d}s\right). \tag{3.5.4}$$

Proof. We start by decomposing L_0 in the (∂_t,∂_i) basis:

$$L_0 = L_0^t\partial_t + L_0^i\partial_i.$$

Recalling (3.1.5) we have

$$\begin{aligned} L_0^t &= -g_0^{tt}\partial_t u_0 - g_0^{it}\partial_i u_0, \\ L_0^i &= -g_0^{ti}\partial_t u_0 - g_0^{ji}\partial_j u_0. \end{aligned}$$

Since u_0 satisfies the eikonal equation (3.1.3) we have

$$g_0^{tt}(\partial_t u_0)^2 + 2g_0^{ti}\partial_t u_0\partial_i u_0 + |\nabla u_0|^2_{\bar{g}_0} = 0$$

Taking ε small enough in (3.1.8) and (3.1.9) then ensures that

$$\left|\frac{1}{L_0^t}\right| + |L_0^\mu| + |\partial L_0^\mu| \leq C(C_0) \tag{3.5.5}$$

on $[0,1]\times\mathbb{R}^3$. We now prove the energy estimate. After multiplication by h, the equation (3.5.3) can be rewritten as

$$\frac{1}{2}\partial_t h^2 = \frac{fh}{L_0^t} - \frac{1}{2}\frac{L_0^i}{L_0^t}\partial_i h^2. \tag{3.5.6}$$

For $s\in[0,T]$ we integrate (3.5.6) on Σ_s with respect to the usual Lebesgue measure and integrate by part in the last integral to obtain

$$\frac{1}{2}\int_{\Sigma_s}\partial_t h^2\mathrm{d}x = \int_{\Sigma_s}\frac{fh}{L_0^t}\mathrm{d}x + \frac{1}{2}\int_{\Sigma_s}\partial_i\left(\frac{L_0^i}{L_0^t}\right)h^2\mathrm{d}x. \tag{3.5.7}$$

In the integral in the LHS, we recognize the time derivative of an energy for h. Therefore (3.5.7) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\|h\|^2_{L^2}(s) = 2\int_{\Sigma_s}\frac{fh}{L_0^t}\mathrm{d}x + \int_{\Sigma_s}\partial_i\left(\frac{L_0^i}{L_0^t}\right)h^2\mathrm{d}x.$$

Using now the Cauchy-Schwarz inequality for the first integral in the RHS and (3.5.5) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|h\|^2_{L^2}(s) \leq C(C_0)\left(\|h\|^2_{L^2}(s) + \|f\|^2_{L^2}(s)\right)$$

Integrating this inequality on $[0,t]$ for $t\leq T$ concludes the proof. ☐

3.5.2 Solving for $F^{(1)}$, $F^{(2,1)}$ and $F^{(2,2)}$

Equations (3.4.4)-(3.4.5)-(3.4.6) have a triangular structure:

- (3.4.4) only depends on the background quantities through the coefficients of the operator $\mathscr{L}_0$,
- the RHS of (3.4.5) depends in addition on $F^{(1)}$,
- the RHS of (3.4.6) depends on the background quantities, $F^{(1)}$ and $F^{(2,1)}$.

Moreover, thanks to the support properties stated in Corollary 3.4.1, we obtain that the background perturbations are supported in $\{|x| \leq C_{\text{supp}} R\}$.

The conclusion of this discussion is that the estimates stated in Theorem 3.5.1 follow directly from Lemma 3.5.1, the estimates of the RHS of (3.4.5)-(3.4.6) we prove in the following lemma and the initial regularity stated in Corollary 3.4.1, see (3.4.43).

Lemma 3.5.2. *We have*

$$\|RHS\ of\ (3.4.5)\|_{H^{N-2}} \lesssim C(C_0) \left\| F^{(1)} \right\|_{H^N}, \tag{3.5.8}$$

$$\|RHS\ of\ (3.4.6)\|_{H^{N-2}} \lesssim C(C_0) \left(\left\| F^{(1)} \right\|_{H^N} + \left\| F^{(2,1)} \right\|_{H^{N-2}} \right). \tag{3.5.9}$$

Proof. The estimates (3.5.8) and (3.5.9) follow from (3.3.13), (3.3.37), (3.3.31) and (3.3.32) and the background regularity stated in Section 3.1.2. □

3.5.3 Algebraic properties of $F^{(1)}$

In this section we prove that $F^{(1)}$ satisfies (3.1.26) and (3.1.27). To this purpose, we define

$$\underline{\zeta}_A = \frac{1}{2} g(\mathbf{D}_{L_0} \underline{L}_0, e_A).$$

Thanks to the properties of the background null frame, we obtain the following decomposition of $\mathbf{D}_{L_0} X$ for X a vector field of the null frame:

$$\mathbf{D}_{L_0} L_0 = 0, \tag{3.5.10}$$

$$\mathbf{D}_{L_0} \underline{L}_0 = 2\underline{\zeta}_A e_A, \tag{3.5.11}$$

$$\mathbf{D}_{L_0} e_A = \underline{\zeta}_A L_0 + \nabla\!\!\!\!/_{L_0} e_A, , \tag{3.5.12}$$

where $\nabla\!\!\!\!/_{L_0} X$ is the projection of $\mathbf{D}_{L_0} X$ onto $TP_{t,u}$. Note that the regularity of the background stated in Section 3.1.2 is enough to ensure that $\underline{\zeta}_A$ as well as $g_0(\nabla\!\!\!\!/_{L_0} e_A, e_B)$ are bounded on $\mathcal{M}$.

3.5.3.1 Propagation of the first polarization

We state a general commutation result.

Lemma 3.5.3. *Let T be a symmetric 2-tensor. If* $\operatorname{Pol}(T) \restriction \Sigma_0 = 0$ *and* $\operatorname{Pol}(\mathscr{L}_0 T) = 0$ *on* $\mathcal{M}$*, then*

$$\operatorname{Pol}(T) = 0$$

on $\mathcal{M}$.

Proof. We start by using the structure equations of the background null frame to compute a transport equation for $\mathrm{Pol}(T)$. Using (3.5.10) we have

$$L_0 T_{L_0 L_0} = (\mathbf{D}_{L_0} T)_{L_0 L_0} + 2T_{L_0 \alpha} \mathbf{D}_{L_0} L_0^\alpha = (\mathbf{D}_{L_0} T)_{L_0 L_0}.$$

Using in addition (3.5.12) we obtain

$$\begin{aligned} L_0 T_{L_0 A} &= (\mathbf{D}_{L_0} T)_{L_0 A} + T_{L_0 \alpha} \mathbf{D}_{L_0} e_A^\alpha + T_{A\alpha} \mathbf{D}_{L_0} L_0^\alpha \\ &= (\mathbf{D}_{L_0} T)_{L_0 A} + \underline{\zeta}_A T_{L_0 L_0} + \delta^{BC} g_0(\nabla\!\!\!\!/_{L_0} e_A, e_B) T_{L_0 C}. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} L_0 \left(\delta^{AB} T_{AB}\right) &= \delta^{AB} (\mathbf{D}_{L_0} T)_{AB} + \delta^{AB} T_{A\alpha} \mathbf{D}_{L_0} e_B^\alpha \\ &= \delta^{AB} (\mathbf{D}_{L_0} T)_{AB} + 2\delta^{AB} \underline{\zeta}_A T_{L_0 B} \end{aligned}$$

where we also use

$$\delta^{AB} \delta^{CD} T_{AC} g_0(\nabla\!\!\!\!/_{L_0} e_B, e_D) = 0$$

which holds because T is symmetric and $g_0(\nabla\!\!\!\!/_{L_0} \cdot, \cdot)$ is antisymmetric. Recalling (3.4.1), this gives the following identities

$$\begin{aligned} \mathscr{L}_0 T_{L_0 L_0} &= (\mathscr{L}_0 T)_{L_0 L_0}, \\ \mathscr{L}_0 T_{L_0 A} &= (\mathscr{L}_0 T)_{L_0 A} - 2\underline{\zeta}_A T_{L_0 L_0} - 2\delta^{BC} g_0(\nabla\!\!\!\!/_{L_0} e_A, e_B) T_{L_0 C}, \\ \mathscr{L}_0 \left(\delta^{AB} T_{AB}\right) &= \delta^{AB} (\mathscr{L}_0 T)_{AB} - 4\delta^{AB} \underline{\zeta}_A T_{B L_0}. \end{aligned}$$

Now, thanks to (3.2.9)-(3.2.10)-(3.2.11), the assumption $\mathrm{Pol}\,(\mathscr{L}_0 T) = 0$ can be used to rewrite the previous system:

$$\begin{aligned} \mathscr{L}_0 \mathrm{Pol}_{L_0}(T) &= 0, \\ \mathscr{L}_0 \mathrm{Pol}_A(T) &= -2\underline{\zeta}_A \mathrm{Pol}_{L_0}(T) - 2\delta^{BC} g_0(\nabla\!\!\!\!/_{L_0} e_A, e_B) \mathrm{Pol}_C(T), \\ \mathscr{L}_0 \mathrm{Pol}_{\underline{L}_0}(T) &= -4\delta^{AB} \underline{\zeta}_A \mathrm{Pol}_B(T). \end{aligned} \tag{3.5.13}$$

This system is a homogeneous and linear transport system for the null components of $\mathrm{Pol}\,(T)$. If we define the energy

$$E = \sum_{X \in \{L_0, \underline{L}_0, e_1, e_2\}} \|\mathrm{Pol}_X(T)\|_{L^2}^2$$

then Lemma 3.5.1 applied to the system (3.5.13) and $\mathrm{Pol}\,(T) \restriction \Sigma_0 = 0$ implies that

$$E(t) \lesssim \int_0^t E(s) \mathrm{d}s$$

for all $t \in [0, 1]$ and with a constant depending only on the background null frame. Gronwall's inequality then implies that $E(t) = 0$ on the whole spacetime, which concludes the proof. □

When applied to $F^{(1)}$ this lemma gives the following.

Corollary 3.5.1. *We have*

$$\mathrm{Pol}\left(F^{(1)}\right) = 0 \tag{3.5.14}$$

and

$$F^{(1)}_{L_0 \underline{L}_0} = 0. \tag{3.5.15}$$

Proof. The equation (3.4.4) directly implies that $\mathrm{Pol}\left(\mathscr{L}_0 F^{(1)}\right) = 0$ and $\mathrm{Pol}\left(F^{(1)}\right) \upharpoonright \Sigma_0 = 0$ holds (see (3.4.45)). We can thus apply Lemma 3.5.3 to obtain (3.5.14). The proof of (3.5.15) is similar: using the null structure equations (3.5.10) and (3.5.11) we obtain

$$\mathbf{D}_{L_0} F^{(1)}_{L_0 \underline{L}_0} = \left(\mathbf{D}_{L_0} F^{(1)}\right)_{L_0 \underline{L}_0} + 2\delta^{AB} \underline{\zeta}_A F^{(1)}_{L_0 B} = \left(\mathbf{D}_{L_0} F^{(1)}\right)_{L_0 \underline{L}_0}$$

where we used $F^{(1)}_{L_0 A} = 0$ (which follows from (3.5.14)). Together with (3.4.4) this last computation implies that $F^{(1)}_{L_0 \underline{L}_0}$ satisfies

$$\mathscr{L}_0 F^{(1)}_{L_0 \underline{L}_0} = 0.$$

Since $F^{(1)}_{L_0 \underline{L}_0} \upharpoonright \Sigma_0 = 0$ (see (3.4.22) and (3.4.21)), this implies (3.5.15) if one uses the energy estimate of Lemma 3.5.1 and Gronwall's inequality (see the end of the proof of the previous lemma for the details). □

Remark 3.5.1. *This lemma shows the consistency of the computations of Section 3.3.2, which were made under the assumptions that (3.5.14) and (3.5.15) hold. Note that (3.5.14) is actually the same as (3.1.26) and that (3.5.14) and (3.5.15) are equivalent to* $F^{(1)}_{L_0 \alpha} = 0$ *and* $\mathrm{tr}_{g_0} F^{(1)} = 0$.

The proof of Lemma 3.5.3 shows that we can obtain a transport equation on the polarization tensors $\mathrm{Pol}(T)$ of a tensor if we know $\mathrm{Pol}(\mathscr{L}_0 T)$. In the case of $F^{(1)}$ it is obviously the case thanks to (3.4.4), and we easily obtained the first part of Corollary 3.5.1. However, the transport equations for $F^{(2,i)}$, i.e (3.4.5) and (3.4.6), are not as simple as (3.4.4) and we can't hope to compute $\mathrm{Pol}\left(\mathscr{L}_0 F^{(2,i)}\right)$ easily. A second difficulty comes from the fact that the polarization conditions for $F^{(2,i)}$ are not simply $\mathrm{Pol}\left(F^{(2,i)}\right) = 0$ but rather involve non-linear expressions depending on $F^{(1)}$ (see (3.4.12) and (3.4.13)). For these two reasons, the propagation of the second polarization, i.e the propagation of the polarization conditions satisfied by $F^{(2,i)}$, can't be proved with Lemma 3.5.3 and is postponed to Section 3.7.3.

3.5.3.2 Propagation of the energy

We now prove that the energy of the gravitational wave described by $g^{(1)}$ is propagated by the evolution.

Lemma 3.5.4. *We have*

$$\mathcal{E}\left(F^{(1)}, F^{(1)}\right) = -4F_0^2 \tag{3.5.16}$$

on $\mathcal{M}$.

Proof. Thanks to (3.2.16) and Corollary 3.5.1, we have

$$\mathcal{E}\left(F^{(1)}, F^{(1)}\right) = -\frac{1}{2}\left|F^{(1)}\right|^2_{g_0} = -\left[\left(F^{(1)}_{e_1 e_1}\right)^2 + \left(F^{(1)}_{e_1 e_2}\right)^2\right].$$

We use (3.4.4), $F^{(1)}_{L_0 A} = 0$ (consequence of (3.5.14)), the null structure equation (3.5.12) and the fact that $g_0(\nabla\!\!\!/_{L_0}\cdot,\cdot)$ is antisymmetric to derive a transport equation for the quantities involved:

$$-L_0\left(F^{(1)}_{e_1 e_1}\right)^2 + (\Box_{g_0} u_0)\left(F^{(1)}_{e_1 e_1}\right)^2 = -4F^{(1)}_{e_1 e_1} F^{(1)}_{e_1 e_2} g_0(\nabla\!\!\!/_{L_0} e_1, e_2),$$
$$-L_0\left(F^{(1)}_{e_1 e_2}\right)^2 + (\Box_{g_0} u_0)\left(F^{(1)}_{e_1 e_2}\right)^2 = 2F^{(1)}_{e_1 e_2} F^{(1)}_{e_1 e_1} g_0(\nabla\!\!\!/_{L_0} e_1, e_2) - 2F^{(1)}_{e_1 e_2} F^{(1)}_{e_2 e_2} g_0(\nabla\!\!\!/_{L_0} e_1, e_2).$$

Thanks to $F^{(1)}_{e_2e_2} = -F^{(1)}_{e_1e_1}$ (another consequence of (3.5.14)) the RHS of the second equation becomes $4F^{(1)}_{e_1e_2}F^{(1)}_{e_1e_1}g_0(\nabla_{L_0}e_1, e_2)$. Therefore, we obtain the following transport equation for the energy:

$$-L_0\mathcal{E}\left(F^{(1)}, F^{(1)}\right) + (\Box_{g_0}u_0)\mathcal{E}\left(F^{(1)}, F^{(1)}\right) = 0.$$

Together with (3.1.4) this last equation implies

$$(-L_0 + \Box_{g_0}u_0)\left(\mathcal{E}\left(F^{(1)}, F^{(1)}\right) + 4F_0^2\right) = 0. \tag{3.5.17}$$

Moreover, (3.4.48) implies that $\mathcal{E}\left(F^{(1)}, F^{(1)}\right) + 4F_0^2$ vanishes on Σ_0, equation (3.5.17) then implies (3.5.16). □

3.6 Solving the reduced system

In this section we solve the reduced system (3.4.7)-(3.4.8). This is done by a bootstrap argument, i.e by showing that we can improve *a priori* estimates using the equations.

3.6.1 Bootstrap assumptions

As explained in Section 3.2.2.2, the reduced system presents a loss of derivatives. The addition of the Fourier cut-off $\Pi_\leq$ in the equation for $\mathfrak{F}$ implies that two derivatives of $\mathfrak{F}$ are actually at the level of one derivative of $\mathfrak{h}$, which is consistent with what we would obtain from an energy estimates on the equation for $\mathfrak{h}$. Indeed, by commuting (3.4.7) with ∇^2 (the case of time derivatives is similar) we obtain broadly from (3.B.12)

$$\left\|\nabla^2\mathfrak{F}\right\|_{L^2} \lesssim \int_0^t \left\|\nabla^2\Pi_\leq(\mathfrak{h})\right\|_{L^2} \lesssim \frac{1}{\lambda}\int_0^t \|\nabla\mathfrak{h}\|_{L^2}.$$

Putting this in the energy estimate for $\mathfrak{h}$ would give $\|\nabla\mathfrak{h}\|_{L^2} \lesssim \frac{1}{\lambda}T$ with T the time of existence. By compensating the largeness of $\frac{1}{\lambda}$ by the smallness of T would prove well-posedness for the system (3.4.7)-(3.4.8) on $[0, T_\lambda] \times \mathbb{R}^3$ for $T_\lambda \longrightarrow 0$ when λ tends to 0. This is far from we want since it doesn't allow us to consider the high-frequency limit $\lambda \longrightarrow 0$ of g_λ. For this we need a time of existence independent of λ. In other words, the addition of $\Pi_\leq$ remove the loss of derivatives but does not remove the "loss of λ". This discussion explains our choice of bootstrap assumptions on $\mathfrak{F}$ and $\mathfrak{h}$:

- $\mathfrak{F}$ is supported in $J_0^+(\{|x| \leq R\})$ and satisfies

$$\|\mathfrak{F}\|_{L^2} + \max_{r\in[\![1,6]\!]}\lambda^{r-1}\|\nabla^r\mathfrak{F}\|_{L^2} + \max_{r\in[\![0,5]\!]}\lambda^r\|\partial_t\nabla^r\mathfrak{F}\|_{L^2} + \max_{r\in[\![0,4]\!]}\lambda^{r+1}\left\|\partial_t^2\nabla^r\mathfrak{F}\right\|_{L^2} \leq A_1\varepsilon, \tag{BA1}$$

$$\max_{r\in[\![0,4]\!]}\lambda^r\left\|\nabla^r\tilde{\Box}_g\mathfrak{F}\right\|_{L^2} \leq A_2\varepsilon, \tag{BA2}$$

- $\mathfrak{h}$ satisfies

$$\max_{r\in[\![0,4]\!]}\lambda^r\left(\|\partial_t\nabla^r\mathfrak{h}\|_{L^2_{\delta+1+r}} + \|\nabla\nabla^r\mathfrak{h}\|_{L^2_{\delta+1+r}}\right) + \max_{r\in[\![0,3]\!]}\lambda^{r+1}\left\|\partial_t^2\nabla^r\mathfrak{h}\right\|_{L^2_{\delta+2+r}} \leq A_3\varepsilon. \tag{BA3}$$

These bootstrap assumptions solve the "loss of λ" since (**BA2**) implies that $\tilde{\Box}_g\mathfrak{F}$ is bounded by λ^0 in L^2 while (**BA1**) implies boundedness by $\frac{1}{\lambda}$. Recovering (**BA2**) is the main challenge we face. Note that the support property of $\mathfrak{F}$ just follows from the transport equation it satisfies (3.4.7), the fact that its RHS is supported in $J_0^+(\{|x|\leq R\})$ and that $\mathfrak{F}\restriction\Sigma_0=0$.

As explained above, a local in time solution to (3.4.7)-(3.4.8) exists *a priori* and by taking the constants A_i large enough (compared to C_{in} of Corollary 3.4.1) we can assume that the set of times t such that (**BA1**)-(**BA2**)-(**BA3**) are satisfied on $[0,t]$ is non-empty. We define T to be the supremum of those times. We have $0<T\leq 1$ but *a priori* T depends on λ. In the next sections, we prove that actually (**BA1**)-(**BA2**)-(**BA3**) holds with better constant on $[0,T]$. This will contradict the definition of T and prove that a solution to (3.4.7)-(3.4.8) exists and satisfy (**BA1**)-(**BA2**)-(**BA3**) on $[0,1]$, that is the interval of existence of g_0, $F^{(1)}$, $F^{(2,1)}$ and $F^{(2,2)}$.

In the sequel, we will always precise the dependence on λ of our estimates. For X a numerical quantity, the notation $C(X)$ will denote any function of X. The symbol $\lesssim$ will denote $\leq C$ for C depending only on δ, R and other irrelevant numerical constants. We already assume that $A_1\ll A_2\ll A_3$ and that $A_i\gg C(C_0)$ for all the $C(C_0)$ that will appear in the proofs.

Remark 3.6.1. *Since λ can be chosen as small as we want, the bootstrap assumptions* (**BA1**) *and* (**BA3**) *imply that $\|X\|_{H^s}\lesssim\|\nabla^sX\|_{L^2}$ for $X\in\{\mathfrak{F},\mathfrak{h}\}$ (for $\mathfrak{h}$ this holds with the appropriate weights at spacelike infinity), i.e that the main term in each H^s norm is the norm of the top derivative. Therefore in the sequel, when using (weighted) Sobolev embedding we will only write down the norm of the top derivative.*

3.6.2 First consequences

We give here the first consequences of the bootstrap assumptions (**BA1**)-(**BA2**)-(**BA3**). Note first that the inequality

$$\|\mathfrak{h}\|_{L^2_{\delta+1}}(t)\lesssim\|\mathfrak{h}\|_{L^2_{\delta+1}}(0)+\int_0^t\|\partial_t\mathfrak{h}\|_{L^2_{\delta+1}}\mathrm{d}s$$

together with (**BA3**) implies $\|\mathfrak{h}\|_{L^2_{\delta+1}}\lesssim C_{\mathrm{in}}\varepsilon+TA_3\varepsilon$. Since $T\leq1$ and A_3 is assumed to be larger than C_{in} we obtain

$$\|\mathfrak{h}\|_{L^2_{\delta+1}}\lesssim A_3\varepsilon. \tag{3.6.1}$$

We now deduce from the bootstrap assumptions some estimates for the metric g. We introduce the following split of the metric g:

$$g=g_{\mathrm{BG}}+\tilde{g} \tag{3.6.2}$$

where

$$g_{\mathrm{BG}}=g_0+\lambda g^{(1)}+\lambda^2\left(F^{(2,1)}\sin\left(\frac{u_0}{\lambda}\right)+F^{(2,2)}\cos\left(\frac{2u_0}{\lambda}\right)\right)+\lambda^3g^{(3,\mathrm{BG})}, \tag{3.6.3}$$

$$\tilde{g}=\lambda^2\left(\mathfrak{F}\sin\left(\frac{u_0}{\lambda}\right)+\mathfrak{h}+\lambda g^{(3)}(\mathfrak{F})\right). \tag{3.6.4}$$

Remark 3.6.2. *Note that the term $g^{(3)}(\mathfrak{F})$ in $\tilde{g}$ will be omitted in the sequel (with the exception of Lemma 3.6.5) since it is basically a linear term in $\mathfrak{F}$ (see Lemma 3.4.1) with an extra λ, and thus behaves at least better than $\mathfrak{F}\sin\left(\frac{u_0}{\lambda}\right)$.*

On the one hand, the metric g_{BG} contains the background metric and all the background perturbations. Thanks to Theorem 3.5.1 it is very regular but because of $g^{(1)}$ its behaviour with respect to λ is bad. On the other hand, the metric $\tilde{g}$ is less regular than g_{BG} but satisfies better estimates in terms of λ. The following lemma gathers the required properties of g, g_{BG} and $\tilde{g}$.

Lemma 3.6.1. *The following estimate holds*

$$\max_{r\in[\![0,3]\!]}\lambda^{r-1}\left\|\nabla^r(g-g_0)\right\|_{L^\infty}\lesssim A_3\varepsilon, \tag{3.6.5}$$

$$\left\|\nabla^4(g_{\mathrm{BG}}-g_0)\right\|_{L^\infty}\lesssim C(C_0)\frac{\varepsilon}{\lambda^3}. \tag{3.6.6}$$

Proof. Let $r\leq 3$. We have $g-g_0=g_{\mathrm{BG}}-g_0+\tilde{g}$. For $g_{\mathrm{BG}}-g_0$ we simply use (3.6.3), the estimates of Theorem 3.5.1 and Lemma 3.4.1 to obtain

$$\left\|\nabla^r(g_{\mathrm{BG}}-g_0)\right\|_{L^\infty}\lesssim C(C_0)\frac{\varepsilon}{\lambda^{r-1}}.$$

The proof of (3.6.6) is similar. For $\tilde{g}$ we use the weighted Sobolev embedding $H^2_{\delta+r}\hookrightarrow L^\infty$ of Proposition 2.1.1:

$$\begin{aligned}
\left\|\nabla^r\tilde{g}\right\|_{L^\infty}&\lesssim\lambda^2\left\|\nabla^r\mathfrak{h}\right\|_{L^\infty}+\sum_{r_1+r_2=r}\frac{\left\|\nabla^{r_1}\mathfrak{F}\right\|_{L^\infty}}{\lambda^{r_2-2}}\\
&\lesssim\lambda^2\left\|\nabla^{r+2}\mathfrak{h}\right\|_{L^2_{\delta+r+2}}+\sum_{r_1+r_2=r}\frac{\left\|\nabla^{r_1+2}\mathfrak{F}\right\|_{L^2}}{\lambda^{r_2-2}}\\
&\lesssim\lambda^2\frac{A_3\varepsilon}{\lambda^{r+1}}+\sum_{r_1+r_2=r}\frac{1}{\lambda^{r_2-2}}\times\frac{A_1\varepsilon}{\lambda^{r_1+1}}\\
&\lesssim\frac{A_3\varepsilon}{\lambda^{r-1}}
\end{aligned}$$

where we used (**BA3**), (**BA1**) and $r\leq 3$. □

Also note that the estimates in Lemma 3.6.1 also holds for the inverse of g, which we can also decompose as in (3.6.2). In what follows we won't make any difference between a coefficient of g or a coefficient of its inverse.

We also deduce from the bootstrap assumptions an estimate for $\mathcal{R}(g)$, i.e the term in (3.4.8) containing only non-problematic terms.

Lemma 3.6.2. *The following estimate holds*

$$\max_{r\in[\![0,4]\!]}\lambda^r\left\|\nabla^r\mathcal{R}(g)\right\|_{L^2_{\delta+1+r}}\lesssim C(A_i)\varepsilon^2. \tag{3.6.7}$$

Proof. In terms of expansions in powers of λ, the definition of $\mathcal{R}(g)$ (see (3.4.9)) ensures the correct behaviour, that is $\mathcal{R}(g)=O\left(\lambda^0\right)$ but since it contains oscillating terms one looses a power of λ for each derivatives. This explains why

$$\left\|\nabla^r\mathcal{R}(g)\right\|_{L^2_{\delta+1+r}}\lesssim\frac{1}{\lambda^r}. \tag{3.6.8}$$

In terms of its dependency on the metric coefficients, we don't need to be very precise about the exact expression of $\mathcal{R}(g)$, its only important property is that it does not contain second order derivatives of either $\mathfrak{F}$, $g^{(3)}$ or $\mathfrak{h}$. Indeed:

- the quadratic non-linearity term $P^{(\geq 2)}$ is of the form $g^{-2}\partial g \partial g$,
- the $\partial^2\mathfrak{F}$, $\partial^2 g^{(3)}$ and $\partial^2\mathfrak{h}$ coming from the wave part of the Ricci tensor are already in (3.4.8) and are thus absent from $\tilde{W}^{(\geq 2)}$,
- the $\partial\mathfrak{F}$, $\partial g^{(3)}$ and $\partial\mathfrak{h}$ coming from the H^ρ term in the Ricci tensor are already put into Υ^ρ and therefore absent from $\mathring{H}^\rho$, which implies that the $\partial^2\mathfrak{F}$, $\partial^2 g^{(3)}$ and $\partial^2\mathfrak{h}$ coming from the gauge part are absent from $\mathcal{R}(g)$.

Therefore, the terms in $\mathcal{R}(g)$ can be put into two categories:

1. the purely background terms, i.e depends only on g_0, $F^{(1)}$ or $F^{(2,i)}$ and their first and second derivatives,
2. the non-background terms, depending quadratically on zeroth or first order derivatives of $\mathfrak{F}$ or $\mathfrak{h}$ with background coefficients.

We can bound all of terms from the first category in L^∞ thanks to the background regularity, Theorem 3.5.1 and the fact that $N \geq 10$ (see Remark 3.2.1). Since those terms are all background, the constant appearing in the estimate is of the form $C(C_0)$.

The terms in the second category can all be estimated using the bootstrap assumptions (**BA1**) and (**BA3**) and by bounding the background coefficients in L^∞. We only give the details for the worse term in terms of λ behaviour and spatial support, i.e $\lambda^2 g_0^{-2} (\partial\mathfrak{h})^2$:

$$\left\| \lambda^2 \nabla^r \left(g_0^{-2} (\partial\mathfrak{h})^2 \right) \right\|_{L^2_{\delta+r+1}} \lesssim C(C_0)\lambda^2 \sum_{r_1+r_2=r} \|\partial\nabla^{r_1}\mathfrak{h}\partial\nabla^{r_2}\mathfrak{h}\|_{L^2_{\delta+r+1}}.$$

If $r_1 \leq 2$, we put $\partial\nabla^{r_1}\mathfrak{h}$ in L^∞, i.e we use the product law of weighted Sobolev spaces in Proposition 2.1.1:

$$\begin{aligned}
\|\partial\nabla^{r_1}\mathfrak{h}\partial\nabla^{r_2}\mathfrak{h}\|_{L^2_{\delta+r+1}} &\lesssim \|\partial\nabla^{r_1}\mathfrak{h}\|_{H^2_{\delta+r_1+1}} \|\partial\nabla^{r_2}\mathfrak{h}\|_{L^2_{\delta+r_2+1}} \\
&\lesssim \frac{A_3\varepsilon}{\lambda^{r_1+2}} \times \frac{A_3\varepsilon}{\lambda^{r_2}}.
\end{aligned}$$

If $r_1 \geq 3$, then $r_2 \leq 2$ (since $r \leq 4$) and we proceed similarly. Finally we obtain

$$\left\| \lambda^2 \nabla^r \left(g_0^{-2} (\partial\mathfrak{h})^2 \right) \right\|_{L^2_{\delta+r+1}} \lesssim \frac{C(A_i)\varepsilon^2}{\lambda^r}.$$

□

3.6.3 Estimates for $\mathfrak{F}$

We start by studying $\mathfrak{F}$, which solves the transport equation (3.4.7). For clarity, we simply write $\mathfrak{h}$ instead of $\mathfrak{h}_{L_0L_0}$ appearing in this equation.

Proposition 3.6.1. *The following estimate holds*

$$\|\mathfrak{F}\|_{L^2} \lesssim A_3\varepsilon^2, \tag{3.6.9}$$

$$\max_{r\in[\![1,6]\!]} \lambda^{r-1}\|\nabla^r\mathfrak{F}\|_{L^2} \lesssim \left(\lambda A_1\varepsilon + A_3\varepsilon^2\right), \tag{3.6.10}$$

$$\max_{r\in[\![0,5]\!]} \lambda^r\|\partial_t\nabla^r\mathfrak{F}\|_{L^2} + \max_{r\in[\![0,4]\!]} \lambda^{r+1}\left\|\partial_t^2\nabla^r\mathfrak{F}\right\|_{L^2} \lesssim \left(\lambda A_1\varepsilon + A_3\varepsilon^2\right). \tag{3.6.11}$$

Proof. We want to apply Lemma 3.5.1, so we need to estimate the L^2 norm of the RHS of (3.4.7), which we rewrite

$$L_0\mathfrak{F}_{\alpha\beta} = \frac{1}{2}\Pi_{\leq}(\mathfrak{h})\,F^{(1)}_{\alpha\beta} + \frac{1}{2}(\Box_{g_0}u_0)\mathfrak{F}_{\alpha\beta} + L_0^\mu\Gamma(g_0)^\nu_{\mu(\alpha}\mathfrak{F}_{\nu\beta)}. \tag{3.6.12}$$

The RHS of (3.6.12) is supported in $\{|x|\leq C_{\mathrm{supp}}R\}$ thanks to the support property of $\mathfrak{F}$ and $F^{(1)}$. We estimate all the background quantities (see Theorem 3.5.1) in L^∞ and obtain

$$\begin{aligned}\|\text{RHS of (3.6.12)}\|_{L^2} &\lesssim C(C_0)\left(\varepsilon\|\Pi_{\leq}(\mathfrak{h})\|_{L^2} + \|\mathfrak{F}\|_{L^2}\right)\\ &\lesssim C(C_0)A_3\varepsilon^2 + C(C_0)\|\mathfrak{F}\|_{L^2}\end{aligned}$$

where we forgot about the projector $\Pi_{\leq}$ and used (3.6.1). Therefore, $\mathfrak{F}\restriction\Sigma_0 = 0$ and Lemma 3.5.1 imply that for all $t\in[0,T]$ we have

$$\|\mathfrak{F}\|^2_{L^2}(t) \lesssim C(C_0)A_3^2\varepsilon^4 + C(C_0)\int_0^t \|\mathfrak{F}\|^2_{L^2}\,\mathrm{d}s$$

which gives (3.6.9) after applying Gronwall's inequality.

We now turn to the proof of (3.6.10), for which we need to commute (3.6.12) with spatial derivatives. For $r\in[\![1,6]\!]$ we schematically obtain

$$L_0\left(\nabla^r\mathfrak{F}\right) = [L_0,\nabla^r]\,\mathfrak{F} + \sum_{r_1+r_2=r}\left(\nabla^{r_1}\Pi_{\leq}(\mathfrak{h})\,\nabla^{r_2}F^{(1)} + \nabla^{r_1}\left(\Box_{g_0}u_0 + L_0\Gamma(g_0)\right)\nabla^{r_2}\mathfrak{F}\right). \tag{3.6.13}$$

From Lemma 3.5.1, (3.6.13) and $\mathfrak{F}\restriction\Sigma_0 = 0$ we have for $t\in[0,T]$

$$\|\nabla^r\mathfrak{F}\|^2_{L^2}(t) \lesssim C(C_0)\int_0^t\left(\|[L_0,\nabla^r]\,\mathfrak{F}\|^2_{L^2} + \sum_{r'\leq r}\left(\varepsilon^2\left\|\nabla^{r'}\Pi_{\leq}(\mathfrak{h})\right\|^2_{L^2} + \left\|\nabla^{r'}\mathfrak{F}\right\|^2_{L^2}\right)\right)\mathrm{d}s \tag{3.6.14}$$

where we again estimate all the background quantities in L^∞.

We estimate the commutator $[L_0,\nabla^r]\,\mathfrak{F}$ using the following formula, true for any linear operators:

$$[A,B^n] = \sum_{k=0}^{n-1} B^{n-1-k}[A,B]B^k. \tag{3.6.15}$$

With the usual Leibniz rule this gives

$$|[L_0,\nabla^r]\,\mathfrak{F}| \lesssim \sum_{k=0}^{r-1}\left|\nabla^{r-1-k}\left((\nabla L_0^\alpha)\partial_\alpha\nabla^k\mathfrak{F}\right)\right| \lesssim C(C_0)\sum_{k=0}^{r-1}\left|\partial\nabla^k\mathfrak{F}\right| \tag{3.6.16}$$

where we used the background regularity.

We first treat the case $r=1$. In order to treat the case of a time derivative in (3.6.16), we rewrite the equation satisfied by $\mathfrak{F}$

$$L_0^t \partial_t \mathfrak{F} = L_0^i \nabla \mathfrak{F} + \Pi_{\leq}(\mathfrak{h}) F^{(1)} + (\Box_{g_0} u_0 + L_0 \Gamma(g_0))\mathfrak{F} \tag{3.6.17}$$

which together with (3.6.1) gives $\|\partial \mathfrak{F}\|_{L^2} \lesssim C(C_0)\|\nabla \mathfrak{F}\|_{L^2} + C(C_0)A_3\varepsilon^2$. Therefore, using (3.6.9) and (3.6.1) we obtain from (3.6.14)

$$\begin{aligned}\|\nabla \mathfrak{F}\|_{L^2}^2(t) &\lesssim C(C_0)A_3^2\varepsilon^4 + C(C_0)\int_0^t \left(\varepsilon^2 \|\nabla \Pi_{\leq}(\mathfrak{h})\|_{L^2}^2 + \|\nabla \mathfrak{F}\|_{L^2}^2\right) \mathrm{d}s \\ &\lesssim C(C_0)A_3^2\varepsilon^4 + C(C_0)\int_0^t \|\nabla \mathfrak{F}\|_{L^2}^2 \,\mathrm{d}s\end{aligned}$$

where we also forgot about $\Pi_{\leq}$ and used (**BA3**). Using Gronwall's lemma, this proves (3.6.10) in the case $r=1$.

If now $r \geq 2$, we have from (**BA1**)

$$\begin{aligned}\|[L_0, \nabla^r]\mathfrak{F}\|_{L^2}^2 &\lesssim C(C_0)\sum_{k=0}^{r-2}\left\|\partial \nabla^k \mathfrak{F}\right\|_{L^2}^2 + C(C_0)\left\|\partial \nabla^{r-1}\mathfrak{F}\right\|_{L^2}^2 \\ &\lesssim C(C_0)\frac{A_1^2\varepsilon^2}{\lambda^{2(r-2)}} + C(C_0)\left\|\partial \nabla^{r-1}\mathfrak{F}\right\|_{L^2}^2.\end{aligned}$$

For the second term in this last expression we rewrite the equation for ∇^{r-1}:

$$\begin{aligned}L_0^t \partial_t \nabla^{r-1}\mathfrak{F} &= \left[L_0, \nabla^{r-1}\right]\mathfrak{F} + L_0^i \nabla^r \mathfrak{F} \\ &+ \sum_{r_1+r_2=r-1}\left(\nabla^{r_1}\Pi_{\leq}(\mathfrak{h})\nabla^{r_2}F^{(1)} + \nabla^{r_1}(\Box_{g_0}u_0 + L_0\Gamma(g_0))\nabla^{r_2}\mathfrak{F}\right).\end{aligned} \tag{3.6.18}$$

which thanks to (3.6.16), (**BA1**), (**BA3**) and (3.B.12) gives

$$\left\|\partial \nabla^{r-1}\mathfrak{F}\right\|_{L^2}^2 \lesssim C(C_0)\|\nabla^r \mathfrak{F}\|_{L^2}^2 + C(C_0)\frac{A_3^2\varepsilon^4 + A_1^2\varepsilon^2}{\lambda^{2(r-2)}}. \tag{3.6.19}$$

Therefore, we have estimated the commutator in (3.6.14):

$$\|[L_0, \nabla^r]\mathfrak{F}\|_{L^2}^2 \lesssim C(C_0)\|\nabla^r \mathfrak{F}\|_{L^2}^2 + C(C_0)\frac{\lambda^2 A_3^2\varepsilon^4 + \lambda^2 A_1^2\varepsilon^2}{\lambda^{2(r-1)}}.$$

For the first sum in the time integral in (3.6.14), we distinguish between $r'=r$ and $r' \leq r-1$ and use (3.B.12) and (**BA3**) (and $r-1 \leq 5$):

$$\sum_{r' \leq r}\varepsilon^2\left\|\nabla^{r'}\Pi_{\leq}(\mathfrak{h}_{L_0L_0})\right\|_{L^2}^2 \lesssim \frac{\varepsilon^2}{\lambda^2}\|\nabla^{r-1}\mathfrak{h}\|_{L^2}^2 + \sum_{r' \leq r-1}\varepsilon^2\left\|\nabla^{r'}\mathfrak{h}\right\|_{L^2}^2 \lesssim \frac{A_3^2\varepsilon^4}{\lambda^{2(r-1)}}.$$

For the second sum in the time integral in (3.6.14), we simply use (**BA1**) if $r' \leq r-1$. Putting everything together gives

$$\|\nabla^r \mathfrak{F}\|_{L^2}^2(t) \lesssim C(C_0)\left(\frac{\lambda^2 A_1^2\varepsilon^2}{\lambda^{2(r-1)}} + \frac{A_3^2\varepsilon^4}{\lambda^{2(r-1)}}\right) + C(C_0)\int_0^t \|\nabla^r \mathfrak{F}\|_{L^2}^2 \,\mathrm{d}s$$

which gives (3.6.10) after applying Gronwall's lemma. Now that (3.6.10) is proved, we notice that (3.6.19) implies the first part of (3.6.11). For the second part, we apply ∂_t to (3.6.18) and estimate the result with (3.6.10). □

3.6.4 Estimates for $\tilde{\Box}_g \mathfrak{F}$

In this section, we derive the estimates for $\tilde{\Box}_g \mathfrak{F}$. This is the most crucial part of the proof, where we deal with the loss of derivatives exhibited in Section 3.2.2.2. As explained in Section 3.2.2.3, we start by using the high-frequency character of g to decompose $\tilde{\Box}_g \mathfrak{F}_{\alpha\beta}$ as in

$$\tilde{\Box}_g \mathfrak{F}_{\alpha\beta} = \tilde{\Box}_{g_0} \mathfrak{F}_{\alpha\beta} + (g^{\mu\nu} - g_0^{\mu\nu})\, \partial_\mu \partial_\nu \mathfrak{F}_{\alpha\beta}. \tag{3.6.20}$$

We are going to estimate separately the two terms in (3.6.20).

3.6.4.1 Estimates for $\Box_{g_0} \mathfrak{F}$

In this section, we prove by a finite strong induction argument on the value $r \in [\![0,4]\!]$ that

$$\|\nabla^r \Box_{g_0} \mathfrak{F}\|_{L^2} \lesssim \frac{C(A_i)\varepsilon^2 + A_1 \varepsilon}{\lambda^r}. \tag{3.6.21}$$

The proof of the induction step or the base case are similar. Therefore, we will prove (3.6.21) for a fixed value of r and assume that it holds for all value strictly less than r in $[\![0,4]\!]$. This induction assumption will only be used in the proof of Lemma 3.6.8.

In order to estimate $\nabla^r \Box_{g_0} \mathfrak{F}$, we start by deriving a transport equation from (3.4.7):

$$L_0(\nabla^r \Box_{g_0} \mathfrak{F}) = \nabla^r [L_0, \Box_{g_0}] \mathfrak{F} + [L_0, \nabla^r] \Box_{g_0} \mathfrak{F} + \nabla^r \Box_{g_0} L_0 \mathfrak{F}. \tag{3.6.22}$$

We give names to the three terms in (3.6.22):

$$\begin{aligned} \mathbf{A}_r &:= \nabla^r [L_0, \Box_{g_0}] \mathfrak{F}, \\ \mathbf{B}_r &:= [L_0, \nabla^r] \Box_{g_0} \mathfrak{F}, \\ \mathbf{C}_r &:= \nabla^r \Box_{g_0} L_0 \mathfrak{F}. \end{aligned}$$

Remark 3.6.3. *These notations and the induction argument presented above will allow us to use the triangular structure of the equations*

$$\begin{aligned} L_0(\Box_{g_0} \mathfrak{F}) &= \mathbf{A}_0 + \mathbf{B}_0 + \mathbf{C}_0 \\ L_0(\nabla \Box_{g_0} \mathfrak{F}) &= \mathbf{A}_1 + \mathbf{B}_1 + \mathbf{C}_1 \\ &\vdots \\ L_0(\nabla^r \Box_{g_0} \mathfrak{F}) &= \mathbf{A}_r + \mathbf{B}_r + \mathbf{C}_r. \end{aligned}$$

Indeed, during the proof of Lemma 3.6.8, we will estimate $\mathbf{B}_r$ in terms of the $\mathbf{A}_k$, $\mathbf{B}_k$ and $\mathbf{C}_k$ for $0 \leq k \leq r-1$ (see (3.6.36)).

We start by estimating $\mathbf{A}_r$.

Lemma 3.6.3. *We have*

$$\|\mathbf{A}_r\|_{L^2} \lesssim \frac{A_1 \varepsilon + A_3 \varepsilon^2}{\lambda^r} + \|\nabla^r \Box_{g_0} \mathfrak{F}\|_{L^2}.$$

Proof. We start with the case $r \geq 1$. We apply (3.2.26) to $\mathfrak{F}$:

$$\|\nabla^r [L_0, \Box_{g_0}] \mathfrak{F}\|_{L^2} \lesssim \|\nabla^r \partial L_0 \mathfrak{F}\|_{L^2} + \|\nabla^r \Box_{g_0} \mathfrak{F}\|_{L^2} + \|\partial \mathfrak{F}\|_{H^r} + \left\|\partial^2 \mathfrak{F}\right\|_{H^{r-1}}.$$

Thanks to (**BA1**) and $r \leq 4$ we have

$$\|\partial \mathfrak{F}\|_{H^r} + \left\|\partial^2 \mathfrak{F}\right\|_{H^{r-1}} \lesssim \frac{A_1 \varepsilon}{\lambda^r}.$$

Since the equation (3.4.7) schematically reads $L_0 \mathfrak{F} = \mathfrak{F} + \mathfrak{h} F^{(1)}$ we also have

$$\begin{aligned}\|\nabla^r \partial L_0 \mathfrak{F}\|_{L^2} &\lesssim \sum_{k=0}^{r} \left(\left\|\partial \nabla^k \mathfrak{F}\right\|_{L^2} + \varepsilon \left\|\partial \nabla^k \mathfrak{h}\right\|_{L^2} \right) \\ &\lesssim \frac{A_1 \varepsilon + A_3 \varepsilon^2}{\lambda^r}\end{aligned}$$

where we used (**BA1**), (**BA3**) and $r \leq 4$. This concludes the proof of the lemma in the case $r \geq 1$. The case $r = 0$ is treated similarly with (3.2.25) instead of (3.2.26). □

We now estimate $\mathbf{C}_r$. Using (3.4.7) we see that

$$|\mathbf{C}_r| \lesssim \left|\nabla^r \tilde{\Box}_{g_0} \Pi_{\leq} (\mathfrak{h}) F^{(1)}\right| + |\nabla^r \Box_{g_0} \mathfrak{F}| + |\mathcal{Q}| \tag{3.6.23}$$

where $\mathcal{Q}$ contains the terms where at most $r+1$ derivatives (out of the $r+2$ involved in $\nabla^r \tilde{\Box}_{g_0}$) hit either $\mathfrak{F}$ or $\mathfrak{h}$. In this case, the last derivative must therefore hit some background quantity. This implies that $\mathcal{Q}$ contains only lower order terms and using the background regularity we easily obtain

$$\begin{aligned}\|\mathcal{Q}\|_{L^2} &\lesssim \sum_{r' \leq r-1} \left(\left\|\partial^2 \nabla^{r'} \mathfrak{F}\right\|_{L^2} + \varepsilon \left\|\partial^2 \nabla^{r'} \mathfrak{h}\right\|_{L^2} \right) \\ &\lesssim \frac{A_1 \varepsilon + A_3 \varepsilon^2}{\lambda^r}\end{aligned} \tag{3.6.24}$$

where the ε in front of $\left\|\partial^2 \nabla^{r'} \mathfrak{h}\right\|_{L^2}$ comes from $F^{(1)}$ and where we used (**BA1**), (**BA3**) and $r - 1 \leq 3$.

We now turn to the estimates of the first term in (3.6.23), i.e $\nabla^r \tilde{\Box}_{g_0} \Pi_{\leq} (\mathfrak{h})$. Forgetting about $\Pi_{\leq}$ and g_0, this term is of the form $\Box \mathfrak{h}$, for which we would like to use (3.4.8). In order to do this, we need first to commute $\tilde{\Box}_{g_0}$ and $\Pi_{\leq}$ and second to replace $\tilde{\Box}_{g_0}$ by $\tilde{\Box}_g$. This give the following decomposition:

$$\nabla^r \tilde{\Box}_{g_0} \Pi_{\leq} (\mathfrak{h}) = \nabla^r \left[\tilde{\Box}_{g_0}, \Pi_{\leq}\right] \mathfrak{h} + \nabla^r \Pi_{\leq} \left(\tilde{\Box}_g \mathfrak{h}\right) - \nabla^r \Pi_{\leq} \left(\tilde{\Box}_{g-g_0} \mathfrak{h}\right). \tag{3.6.25}$$

The lemmas 3.6.4, 3.6.5 and 3.6.6 below estimate the three different terms in the decomposition (3.6.25). Since $\nabla^r \tilde{\Box}_{g_0} \Pi_{\leq} (\mathfrak{h})$ is multiplied by $F^{(1)}$ in (3.6.23), giving an extra ε, we don't need to be very precise in terms of the bootstrap constants A_i. For the same reason, the support property of $F^{(1)}$ implies that we can forget about the weight at spacelike infinity for $\mathfrak{h}$ or g (except in Lemma 3.6.5 below).

Lemma 3.6.4. *We have*

$$\left\|\nabla^r \left[\tilde{\Box}_{g_0}, \Pi_{\leq}\right] \mathfrak{h}\right\|_{L^2} \lesssim \frac{A_3 \varepsilon}{\lambda^r}.$$

Proof. Thanks to $[\nabla^r, \Pi_{\leq}] = 0$ and to the Leibniz rule we obtain

$$\left\|\nabla^r \left[\tilde{\Box}_{g_0}, \Pi_{\leq}\right] \mathfrak{h}\right\|_{L^2} \lesssim \sum_{r_1 + r_2 = r} \left\|\left[\nabla^{r_1} g_0 \partial^2 \nabla^{r_2}, \Pi_{\leq}\right] \mathfrak{h}\right\|_{L^2}.$$

We first study the case where $r_2 \leq 3$. We use $\left[\partial^2\nabla^{r_2}, \Pi_{\leq}\right] = 0$ and (3.B.14)

$$\begin{aligned}\left\|\left[\nabla^{r_1} g_0 \partial^2\nabla^{r_2}, \Pi_{\leq}\right]\mathfrak{h}\right\|_{L^2} &= \left\|\left[\nabla^{r_1} g_0, \Pi_{\leq}\right]\partial^2\nabla^{r_2}\mathfrak{h}\right\|_{L^2} \\ &\lesssim \lambda\left\|\nabla^{r_1+1}g_0\right\|_{L^\infty}\left\|\partial^2\nabla^{r_2}\mathfrak{h}\right\|_{L^2} \\ &\lesssim C(C_0)\lambda \times \frac{A_3\varepsilon}{\lambda^{r_2+1}} \\ &\lesssim C(C_0)\frac{A_3\varepsilon}{\lambda^r}\end{aligned}$$

where we used the background regularity and (**BA3**) (with $r_2 \leq 3$). If now $r_2 = 4$, then $r_1 = 0$ and $r = 4$. Since we can't estimate $\partial^2\nabla^4\mathfrak{h}$, we need to gain one spatial derivative, and for this purpose we use (3.2.30) instead of (3.B.14) to obtain

$$\begin{aligned}\left\|\left[g_0\partial^2\nabla^4, \Pi_{\leq}\right]\mathfrak{h}\right\|_{L^2} &= \left\|\left[g_0, \Pi_{\leq}\right]\partial^2\nabla^4\mathfrak{h}\right\|_{L^2} \\ &\lesssim C(C_0)\left\|\partial^2\nabla^3\mathfrak{h}\right\|_{L^2} \\ &\lesssim C(C_0)\frac{A_3\varepsilon}{\lambda^4}\end{aligned}$$

where we also used the background regularity and (**BA3**). $\square$

In the next lemma, we estimate the term $\nabla^r\Pi_{\leq}\left(\tilde{\Box}_g\mathfrak{h}\right)$ in (3.6.25) and therefore we need to estimate the L^2 norm of

$$\nabla^r\Pi_{\leq}\left(\text{RHS of (3.4.8)}\right). \tag{3.6.26}$$

In the forthcoming Section 3.6.5 we will improve (**BA3**) with energy estimates for the wave operator. Therefore, we will apply ∇^r to (3.4.8) and will need an estimate for the *weighted* L^2 norm of (3.6.26) *without* the projector $\Pi_{\leq}$ (in addition to a commutator estimate, see Proposition 3.6.5). This explains why in the next lemma we project ourself and first prove (3.6.27) and then get (3.6.28) as a consequence.

Lemma 3.6.5. *We have*

$$\left\|\nabla^r\tilde{\Box}_g\mathfrak{h}\right\|_{L^2_{\delta+r+1}} \lesssim \frac{C(A_i)\varepsilon^2 + A_2\varepsilon}{\lambda^r} \tag{3.6.27}$$

which in particular implies

$$\left\|\nabla^r\Pi_{\leq}\left(\tilde{\Box}_g\mathfrak{h}\right)\right\|_{L^2} \lesssim C(A_i)\frac{\varepsilon}{\lambda^r}. \tag{3.6.28}$$

Proof. We only need to prove (3.6.27). As explained above, we rely on the equation satisfied by $\mathfrak{h}$, that is (3.4.8). We have

$$\begin{aligned}\left\|\nabla^r\tilde{\Box}_g\mathfrak{h}\right\|_{L^2_{\delta+1+r}} &\lesssim \sum_{r_1+r_2=r}\frac{\left\|\nabla^{r_1}\tilde{\Box}_g\mathfrak{F}\right\|_{L^2}}{\lambda^{r_2}} + \lambda\left\|\nabla^r\tilde{\Box}_g g^{(3)}\right\|_{L^2} \\ &\quad + \frac{\varepsilon}{\lambda}\sum_{r_1+r_2=r}\frac{\left\|\nabla^{r_1}\Pi_{\geq}\left(\mathfrak{h}\right)\right\|_{L^2}}{\lambda^{r_2}} + \left\|\nabla^r\mathcal{R}(g)\right\|_{L^2_{\delta+1+r}} \\ &=: I + II + III + IV\end{aligned} \tag{3.6.29}$$

where the ε in front of $\Pi_{\geq}\left(\mathfrak{h}\right)$ comes from $F^{(1)}$. For I we simply use (**BA2**) to get

$$I \lesssim \frac{A_2\varepsilon}{\lambda^r}. \tag{3.6.30}$$

For II, we first note that $g^{(3)}$ is oscillating and therefore one looses potentially two powers of λ in $\tilde{\Box}_g g^{(3)}$. However this is not the case since if T is a trigonometric function (recall (3.2.4)) and f a scalar function then we have schematically

$$\begin{aligned}\tilde{\Box}_g \left(\lambda \mathrm{T}\left(\frac{u_0}{\lambda}\right) f\right) &= \frac{1}{\lambda} g^{-1}(\mathrm{d}u_0, \mathrm{d}u_0) f + g\partial f + \lambda \tilde{\Box}_g f \\ &= g\partial f + \lambda\left(\tilde{\Box}_g f + gf\right) \qquad (3.6.31)\end{aligned}$$

where we used $g^{-1}(\mathrm{d}u_0, \mathrm{d}u_0) = O\left(\lambda^2\right)$ (see (3.3.7)) and did not write the trigonometric functions. Therefore, we use the decomposition of Lemma 3.4.1 and obtain from (3.6.31) on the one hand

$$\lambda \left\| \nabla^r \tilde{\Box}_g g^{(3,\mathrm{BG})} \right\|_{L^2} \lesssim \frac{C(A_i)\varepsilon^2 + \varepsilon}{\lambda^r}$$

and on the other hand

$$\lambda \left\| \nabla^r \tilde{\Box}_g g^{(3)}(\mathfrak{F}) \right\|_{L^2} \lesssim \sum_{r_1+r_2=r} \| \nabla^{r_1} g \partial \nabla^{r_2} \mathfrak{F} \|_{L^2} + \lambda \left\| \nabla^r \tilde{\Box}_g \mathfrak{F} \right\|_{L^2}$$

where for the sake of clarity we replaced each $g^{(3,\mathrm{T})}(\mathfrak{F})$ by $\mathfrak{F}$ (see (3.4.20)). We study the sum. If $r_1 \leq 3$, then we use (3.6.5) and $\|g_0\|_{L^\infty} \lesssim 1$ and (**BA1**) to write

$$\| \nabla^{r_1} g \partial \nabla^{r_2} \mathfrak{F} \|_{L^2} \lesssim \frac{C(A_i)\varepsilon^2 + A_2\varepsilon}{\lambda^{r-1}}.$$

If $r_1 = 4$, then we use the decomposition (3.6.2) and write

$$\begin{aligned}\left\| \nabla^4 g \partial \mathfrak{F} \right\|_{L^2} &\lesssim \left\| \nabla^4 g_{\mathrm{BG}} \partial \mathfrak{F} \right\|_{L^2} + \left\| \nabla^4 \tilde{g} \partial \mathfrak{F} \right\|_{L^2} \\ &\lesssim \left\| \nabla^4 g_{\mathrm{BG}} \right\|_{L^\infty} \| \partial \mathfrak{F} \|_{L^2} + \left\| \nabla^4 \tilde{g} \right\|_{L^2} \| \partial \mathfrak{F} \|_{L^\infty} \\ &\lesssim \frac{A_1 \varepsilon^2}{\lambda^3} + \frac{C(A_i)\varepsilon^2}{\lambda^4}\end{aligned}$$

where we used (**BA1**). Using in addition (**BA2**) we obtain

$$\lambda \left\| \nabla^r \tilde{\Box}_g g^{(3)}(\mathfrak{F}) \right\|_{L^2} \lesssim \frac{C(A_i)\varepsilon^2 + A_2\varepsilon}{\lambda^r}.$$

We have proved that

$$II \lesssim \frac{C(A_i)\varepsilon^2 + A_2\varepsilon}{\lambda^r}. \qquad (3.6.32)$$

For III, we use (3.B.13) to obtain

$$III \lesssim \varepsilon \sum_{r_1+r_2=r} \frac{\| \nabla \nabla^{r_1} \mathfrak{h} \|_{L^2}}{\lambda^{r_2}} \lesssim \frac{A_3 \varepsilon^2}{\lambda^r} \qquad (3.6.33)$$

where we used (**BA3**) and $r_1 \leq 4$. For IV, we simply use (3.6.7). Together with (3.6.30), (3.6.32) and (3.6.33) this concludes the proof. □

Lemma 3.6.6. *We have*

$$\| \nabla^r \Pi_{\leq} (\Box_{g-g_0} \mathfrak{h}) \|_{L^2} \lesssim C(A_i) \frac{\varepsilon^2}{\lambda^r}.$$

Proof. We first treat the case $r=0$. We forget about the projector $\Pi_{\leq}$ and use (3.6.5) and (**BA3**) to obtain

$$\begin{aligned}\left\|\Pi_{\leq}\left((g-g_0)\,\partial^2\mathfrak{h}\right)\right\|_{L^2} &\lesssim \|g-g_0\|_{L^\infty}\left\|\partial^2\mathfrak{h}\right\|_{L^2}\\ &\lesssim \lambda A_3\varepsilon\times\frac{A_3\varepsilon}{\lambda}\\ &\lesssim C(A_i)\varepsilon^2.\end{aligned}$$

If $r\geq 1$ we use (3.B.12) to get rid of one derivative and the Leibniz rule to obtain

$$\begin{aligned}\left\|\nabla^r\Pi_{\leq}\left((g-g_0)\,\partial^2\mathfrak{h}\right)\right\|_{L^2} &\lesssim \frac{1}{\lambda}\left\|\nabla^{r-1}\left((g-g_0)\,\partial^2\mathfrak{h}\right)\right\|_{L^2}\\ &\lesssim \frac{1}{\lambda}\sum_{r_1+r_2=r-1}\|\nabla^{r_1}(g-g_0)\|_{L^\infty}\left\|\partial^2\nabla^{r_2}\mathfrak{h}\right\|_{L^2}\\ &\lesssim \frac{1}{\lambda}\times\frac{A_3\varepsilon}{\lambda^{r_1-1}}\times\frac{A_3\varepsilon}{\lambda^{r_2+1}}\\ &\lesssim C(A_i)\frac{\varepsilon^2}{\lambda^r}\end{aligned}$$

where we again use (3.6.5) in addition to (**BA3**) and the fact that $r_i\leq 3$. □

Putting together Lemmas 3.6.4, 3.6.5 and 3.6.6 and the estimate (3.6.24) we obtain

Lemma 3.6.7. *We have*

$$\|\mathbf{C}_r\|_{L^2}\lesssim\frac{A_1\varepsilon+A_3\varepsilon^2}{\lambda^r}+\|\nabla^r\Box_{g_0}\mathfrak{F}\|_{L^2}.$$

We now estimate $\mathbf{B}_r$.

Lemma 3.6.8. *We have*

$$\|\mathbf{B}_r\|_{L^2}\lesssim\frac{A_1\varepsilon+C(A_i)\varepsilon^2}{\lambda^r}+\|\nabla^r\Box_{g_0}\mathfrak{F}\|_{L^2}.$$

Proof. Note that $\mathbf{B}_0=0$ so we can assume $r\geq 1$. We first use (3.6.16) to obtain

$$\|\mathbf{B}_r\|_{L^2}\lesssim\sum_{k=0}^{r-1}\left\|\nabla^{k+1}\Box_{g_0}\mathfrak{F}\right\|_{L^2}+\sum_{k=0}^{r-1}\left\|\partial_t\nabla^k\Box_{g_0}\mathfrak{F}\right\|_{L^2} \tag{3.6.34}$$

where we distinguished between time and spatial derivatives. For the first sum in (3.6.34), we note that if $k\leq r-2$ we have

$$\left\|\nabla^{k+1}\Box_{g_0}\mathfrak{F}\right\|_{L^2}\lesssim\sum_{\ell=0}^{k+1}\left\|\partial\nabla^\ell\mathfrak{F}\right\|_{L^2}\lesssim\frac{A_1\varepsilon}{\lambda^r}$$

where we used (**BA1**). This gives

$$\sum_{k=0}^{r-1}\left\|\nabla^{k+1}\Box_{g_0}\mathfrak{F}\right\|_{L^2}\lesssim\|\nabla^r\Box_{g_0}\mathfrak{F}\|_{L^2}+\frac{A_1\varepsilon}{\lambda^r}. \tag{3.6.35}$$

For the second sum in (3.6.34) we need to proceed differently since we didn't include time derivative of $\Box_{g_0}\mathfrak{F}$ in the bootstrap assumptions. We use the equation satisfied by $\nabla^k\Box_{g_0}\mathfrak{F}$ which schematically reads

$$\partial_t\nabla^k\Box_{g_0}\mathfrak{F}=\nabla^{k+1}\Box_{g_0}\mathfrak{F}+\mathbf{A}_k+\mathbf{B}_k+\mathbf{C}_k$$

Therefore, if we sum these equalities from 0 to $r-1$ and use (3.6.35) again we obtain

$$\|\mathbf{B}_r\|_{L^2} \lesssim \|\nabla^r \Box_{g_0} \mathfrak{F}\|_{L^2} + \frac{A_1\varepsilon}{\lambda^r} + \sum_{k=0}^{r-1} \left(\|\mathbf{A}_k\|_{L^2} + \|\mathbf{B}_k\|_{L^2} + \|\mathbf{C}_k\|_{L^2}\right). \tag{3.6.36}$$

We can now use Lemmas 3.6.3 and 3.6.7:

$$\begin{aligned}\sum_{k=0}^{r-1} \left(\|\mathbf{A}_k\|_{L^2} + \|\mathbf{C}_k\|_{L^2}\right) &\lesssim \sum_{k=0}^{r-1} \left(\frac{A_1\varepsilon + A_3\varepsilon^2}{\lambda^k} + \left\|\nabla^k \Box_{g_0} \mathfrak{F}\right\|_{L^2} \right) \\ &\lesssim \frac{A_1\varepsilon + C(A_i)\varepsilon^2}{\lambda^{r-1}}\end{aligned}$$

where we used the induction assumption (3.6.21) for $\left\|\nabla^k \Box_{g_0} \mathfrak{F}\right\|_{L^2}$ with $k \leq r-1$. Since we can assume $\lambda \leq \varepsilon$ this shows that

$$\sum_{k=0}^{r-1} \left(\|\mathbf{A}_k\|_{L^2} + \|\mathbf{C}_k\|_{L^2}\right) \lesssim \frac{C(A_i)\varepsilon^2}{\lambda^r}.$$

Going back to (3.6.36) this gives

$$\|\mathbf{B}_r\|_{L^2} \lesssim \|\nabla^r \Box_{g_0} \mathfrak{F}\|_{L^2} + \frac{A_1\varepsilon + C(A_i)\varepsilon^2}{\lambda^r} + \sum_{k=0}^{r-1} \|\mathbf{B}_k\|_{L^2}.$$

Thanks to an iterative argument, this shows that

$$\|\mathbf{B}_r\|_{L^2} \lesssim \|\nabla^r \Box_{g_0} \mathfrak{F}\|_{L^2} + \frac{A_1\varepsilon + C(A_i)\varepsilon^2}{\lambda^r}$$

where we again used the induction assumption (3.6.21) for $\left\|\nabla^k \Box_{g_0} \mathfrak{F}\right\|_{L^2}$ with $k \leq r-1$ and the fact that $\mathbf{B}_0 = 0$.

□

Putting everything together, we finally obtain the following proposition.

Proposition 3.6.2. *The following estimate holds*

$$\max_{r\in[\![0,4]\!]} \lambda^r \left\|\nabla^r \tilde{\Box}_{g_0} \mathfrak{F}\right\|_{L^2} \lesssim C(A_i)\varepsilon^2 + A_1\varepsilon. \tag{3.6.37}$$

Proof. We want to apply Lemma 3.5.1 to the equation (3.6.22). For this we estimate its RHS, by putting together Lemmas 3.6.3, 3.6.8 and 3.6.7:

$$\|\text{RHS of (3.6.22)}\|_{L^2} \lesssim \|\nabla^r \Box_{g_0} \mathfrak{F}\|_{L^2} + \frac{C(A_i)\varepsilon^2 + A_1\varepsilon}{\lambda^r}.$$

Lemma 3.5.1 then implies

$$\|\nabla^r \Box_{g_0} \mathfrak{F}\|_{L^2}(t) \lesssim \|\nabla^r \Box_{g_0} \mathfrak{F}\|_{L^2}(0) + \frac{C(A_i)\varepsilon^2 + A_1\varepsilon}{\lambda^r} + \int_0^t \|\nabla^r \Box_{g_0} \mathfrak{F}\|_{L^2} \mathrm{d}s. \tag{3.6.38}$$

Let us now estimate the initial data term in (3.6.38). Since $\mathfrak{F} \restriction \Sigma_0 = 0$, the only cases we need to consider are $\partial_t \nabla^{r+1} \mathfrak{F}$ and $\partial_t^2 \nabla^r \mathfrak{F}$. The initial data for such terms are obtained by differenciating (3.4.32). Using the background regularity this gives

$$\left\|\partial_t \nabla^{r+1} \mathfrak{F}\right\|_{L^2}(0) \lesssim \varepsilon \sum_{r' \leq r+1} \left\|\nabla^{r'} \Pi_{\leq}(\mathfrak{h})\right\|_{L^2}(0)$$

where the ε comes from $F^{(1)}$ in front of $\Pi_{\leq}(\mathfrak{h})$ in (3.4.32). Using $r \leq 4$ and (**BA3**) we obtain

$$\left\|\partial_t \nabla^{r+1}\mathfrak{F}\right\|_{L^2}(0) \lesssim \frac{A_3\varepsilon^2}{\lambda^r}$$

The proof is similar for $\partial_t^2\nabla^r\mathfrak{F}$ and we omit the details.

Going back to (3.6.38) we have proved that

$$\|\nabla^r\Box_{g_0}\mathfrak{F}\|_{L^2} \lesssim \frac{C(A_i)\varepsilon^2 + A_1\varepsilon}{\lambda^r} + \int_0^t \|\nabla^r\Box_{g_0}\mathfrak{F}\|_{L^2}\,\mathrm{d}s.$$

Gronwall's inequality then implies that

$$\|\nabla^r\Box_{g_0}\mathfrak{F}\|_{L^2} \lesssim \frac{C(A_i)\varepsilon^2 + A_1\varepsilon}{\lambda^r}.$$

This concludes the induction proving that (3.6.21) holds for all values of $r \in [\![0,4]\!]$. Since $\Box_{g_0}\mathfrak{F}$ and $\tilde{\Box}_{g_0}\mathfrak{F}$ only differs by lower order terms, this also concludes the proof of (3.6.37). □

3.6.4.2 Estimates for $(g-g_0)\partial^2\mathfrak{F}$

We now estimate the second term in (3.6.20).

Proposition 3.6.3. *For $r \in [\![0,4]\!]$ we have*

$$\max_{r\in[\![0,4]\!]} \lambda^r \left\|\nabla^r\left((g-g_0)\,\partial^2\mathfrak{F}\right)\right\|_{L^2} \lesssim C(A_i)\varepsilon^2. \tag{3.6.39}$$

Proof. We start by noticing that (**BA1**) imply

$$\max_{r\in[\![0,4]\!]} \lambda^{r+1}\left\|\partial^2\nabla^r\mathfrak{F}\right\|_{L^2} \lesssim A_1\varepsilon \tag{3.6.40}$$

which in turn implies

$$\max_{r\in[\![0,2]\!]} \lambda^{r+3}\left\|\partial^2\nabla^r\mathfrak{F}\right\|_{L^\infty} \lesssim A_1\varepsilon \tag{3.6.41}$$

with the usual $H^2 \hookrightarrow L^\infty$ embedding. To prove the proposition, we start with the Leibniz rule:

$$\left\|\nabla^r\left((g-g_0)\,\partial^2\mathfrak{F}\right)\right\|_{L^2} \lesssim \sum_{r_1+r_2=r}\left\|\nabla^{r_1}(g-g_0)\,\partial^2\nabla^{r_2}\mathfrak{F}\right\|_{L^2}.$$

If $r_1 \leq 3$, then we use (3.6.5) and (3.6.40) to obtain

$$\begin{aligned}\left\|\nabla^{r_1}(g-g_0)\,\partial^2\nabla^{r_2}\mathfrak{F}\right\|_{L^2} &\lesssim \|\nabla^{r_1}(g-g_0)\|_{L^\infty}\left\|\partial^2\nabla^{r_2}\mathfrak{F}\right\|_{L^2}\\ &\lesssim \frac{\varepsilon}{\lambda^{r_1-1}} \times \frac{A_1\varepsilon}{\lambda^{r_2+1}}\\ &\lesssim \frac{A_1\varepsilon^2}{\lambda^r}\end{aligned}$$

where we also used the fact that $r_2 \leq 4$.

If now $r_1 = 4$, we have $r = 4$ ans $r_2 = 0$. In this case we use the decomposition (3.6.2) to write

$$\left\|\nabla^4 (g - g_0) \partial^2 \mathfrak{F}\right\|_{L^2} \lesssim \left\|\nabla^4 (g_{\mathrm{BG}} - g_0) \partial^2 \mathfrak{F}\right\|_{L^2} + \left\|\nabla^4 \tilde{g} \partial^2 \mathfrak{F}\right\|_{L^2}. \tag{3.6.42}$$

For the first term in (3.6.42) we put the background quantities in L^∞:

$$\begin{aligned} \left\|\nabla^4 (g_{\mathrm{BG}} - g_0) \partial^2 \mathfrak{F}\right\|_{L^2} &\lesssim \left\|\nabla^4 (g_{\mathrm{BG}} - g_0)\right\|_{L^\infty} \left\|\partial^2 \mathfrak{F}\right\|_{L^2} \\ &\lesssim C(C_0) \frac{\varepsilon}{\lambda^3} \times \frac{A_1 \varepsilon}{\lambda} \\ &\lesssim C(C_0) \frac{A_1 \varepsilon^2}{\lambda^4} \end{aligned} \tag{3.6.43}$$

where we used (3.6.6) and (3.6.40). For the second term in (3.6.42) we use (3.6.4) to decompose it further:

$$\left\|\nabla^4 \tilde{g} \partial^2 \mathfrak{F}\right\|_{L^2} \leq \lambda^2 \left\|\nabla^4 \left(\sin\left(\frac{u_0}{\lambda}\right) \mathfrak{F}\right) \partial^2 \mathfrak{F}\right\|_{L^2} + \lambda^2 \left\|\nabla^4 \mathfrak{h} \partial^2 \mathfrak{F}\right\|_{L^2} + \lambda^3 \left\|\nabla^4 g^{(3)}(\mathfrak{F}) \partial^2 \mathfrak{F}\right\|_{L^2} \tag{3.6.44}$$

Using (**BA3**) and (3.6.41) we first obtain

$$\begin{aligned} \lambda^2 \left\|\nabla^4 \mathfrak{h} \partial^2 \mathfrak{F}\right\|_{L^2} &\lesssim \lambda^2 \left\|\nabla^4 \mathfrak{h}\right\|_{L^2} \left\|\partial^2 \mathfrak{F}\right\|_{L^\infty} \\ &\lesssim \lambda^2 \times \frac{A_3 \varepsilon}{\lambda^3} \times \frac{A_1 \varepsilon}{\lambda^3} \\ &\lesssim \frac{A_1 A_3 \varepsilon^2}{\lambda^4}. \end{aligned} \tag{3.6.45}$$

Now, by using the Leibniz rule on the product $\sin\left(\frac{u_0}{\lambda}\right) \mathfrak{F}$ and putting aside the case where 4 derivatives hit $\sin\left(\frac{u_0}{\lambda}\right)$ we obtain

$$\begin{aligned} \lambda^2 \left\|\nabla^4 \left(\sin\left(\frac{u_0}{\lambda}\right) \mathfrak{F}\right) \partial^2 \mathfrak{F}\right\|_{L^2} &\lesssim \frac{1}{\lambda^2} \left\|\mathfrak{F} \partial^2 \mathfrak{F}\right\|_{L^2} + \sum_{\substack{a+b=4 \\ a \leq 3}} \frac{1}{\lambda^{a-2}} \left\|\nabla^b \mathfrak{F} \partial^2 \mathfrak{F}\right\|_{L^2} \\ &\lesssim \frac{1}{\lambda^2} \left\|\mathfrak{F}\right\|_{L^\infty} \left\|\partial^2 \mathfrak{F}\right\|_{L^2} + \sum_{\substack{a+b=4 \\ a \leq 3}} \frac{1}{\lambda^{a-2}} \left\|\nabla^b \mathfrak{F}\right\|_{L^2} \left\|\partial^2 \mathfrak{F}\right\|_{L^\infty} \\ &\lesssim \frac{1}{\lambda^2} \times \frac{A_1 \varepsilon}{\lambda} \times \frac{A_1 \varepsilon}{\lambda} + \sum_{\substack{a+b=4 \\ a \leq 3}} \frac{1}{\lambda^{a-2}} \times \frac{A_1 \varepsilon}{\lambda^{b-1}} \times \frac{A_1 \varepsilon}{\lambda^3} \\ &\lesssim \frac{A_1^2 \varepsilon^2}{\lambda^4} \end{aligned} \tag{3.6.46}$$

where we used (3.6.40) and (3.6.41). We treat the better term $\lambda^3 \left\|\nabla^4 g^{(3)}(\mathfrak{F}) \partial^2 \mathfrak{F}\right\|_{L^2}$ in (3.6.44) similarly. Putting (3.6.42), (3.6.43), (3.6.45) and (3.6.46) together gives

$$\left\|\nabla^4 (g - g_0) \partial^2 \mathfrak{F}\right\|_{L^2} \lesssim C(A_i) \frac{\varepsilon^2}{\lambda^4}.$$

This concludes the proof of the proposition. □

Together with the decomposition (3.6.20), the results of Propositions 3.6.2 and 3.6.3 finally imply

Proposition 3.6.4. *The following estimate holds*

$$\max_{r \in [\![0,4]\!]} \lambda^r \left\|\nabla^r \tilde{\Box}_g \mathfrak{F}\right\|_{L^2} \lesssim C(A_i) \varepsilon^2 + A_1 \varepsilon.$$

3.6.5 Estimates for $\mathfrak{h}$

We start by deriving a weighted energy estimate for the principal part of the wave operator associated to g, i.e $\tilde{\Box}_g$. For clarity we define

$$E_\sigma(h) = \|\partial_t h\|^2_{L^2_\sigma} + \|\nabla h\|^2_{L^2_\sigma}$$

for h a scalar function and $\sigma \in \mathbb{R}$.

Lemma 3.6.9. *Let h be a scalar function on $[0,T]\times\mathbb{R}^3$, for all $t\in[0,T]$ we have*

$$E_\sigma(h)(t) \lesssim E_\sigma(h)(0) + (1+C(A_i)\varepsilon)\int_0^t E_\sigma(h)(s)\mathrm{d}s + \int_0^t \left\|\tilde{\Box}_g h\right\|^2_{L^2_\sigma}(s)\mathrm{d}s.$$

Proof. Let $w(x) = \left(1+|x|^2\right)^\sigma$. We set $f = \tilde{\Box}_g h$ and multiply this identity by $w\partial_t h$ and integrate over Σ_t for $t\in[0,T]$ with the usual Lebesgue measure:

$$\int_{\Sigma_t} wg^{tt}\partial_t h\partial_t^2 h\mathrm{d}x + 2\int_{\Sigma_t} wg^{ti}\partial_t h\partial_i\partial_t h\mathrm{d}x + \int_{\Sigma_t} wg^{ij}\partial_t h\partial_i\partial_j h\mathrm{d}x = \int_{\Sigma_t} w\partial_t h f\mathrm{d}x.$$

We rewrite the first integral and integrate by parts in the second and third integrals to obtain:

$$\begin{aligned} -\frac{1}{2}\frac{\mathrm{d}E}{\mathrm{d}t} - \int_{\Sigma_t} w\left(\partial_t g^{tt}\right)(\partial_t h)^2\mathrm{d}x - \int_{\Sigma_t}\partial_i\left(wg^{0i}\right)(\partial_t h)^2\mathrm{d}x \\ - \int_{\Sigma_t}\partial_i\left(wg^{ij}\right)\partial_t h\partial_j h\mathrm{d}x + \frac{1}{2}\int_{\Sigma_t} w\left(\partial_t g^{ij}\right)\partial_i h\partial_j h\mathrm{d}x = \int_{\Sigma_t} w\partial_t h f\mathrm{d}x. \end{aligned} \tag{3.6.47}$$

where

$$E(t) = \int_{\Sigma_t} w\left(-g^{tt}(\partial_t h)^2 + g^{ij}\partial_i h\partial_j h\right).$$

Taking ε small enough in (3.1.8) we obtain the existence of a universal constant $C'>0$ such that

$$\frac{1}{C'}E_\sigma(h) \leq E \leq C' E_\sigma(h). \tag{3.6.48}$$

Using the background regularity and the bootstrap assumptions (**BA1**) and (**BA3**) as in the proof of Lemma 3.6.1 we obtain $\|\partial g\|_{L^\infty} \lesssim A_3\varepsilon$. Using in addition $|\nabla w| \lesssim w$ we obtain $|\partial(wg)| \lesssim (1+C(A_i)\varepsilon)\,w$. Going back to (3.6.47), this implies

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}t}(t) &\lesssim C(A_i)\varepsilon\int_{\Sigma_t} w\partial h\partial h\mathrm{d}x + \int_{\Sigma_t} w\partial_t h f\mathrm{d}x \\ &\lesssim (1+C(A_i)\varepsilon)\,E_\sigma(h)(t) + \|f\|^2_{L^2_\sigma}(t) \end{aligned}$$

where we also used the Cauchy-Schwarz inequality. Integrating this inequality from 0 to $t\in[0,T]$ we obtain

$$E(t) \leq E(0) + (1+C(A_i)\varepsilon)\int_0^t E_\sigma(h)(s)\mathrm{d}s + \int_0^t \|f\|^2_{L^2_\sigma}(s)\mathrm{d}s$$

Using (3.6.48) concludes the proof. $\square$

Proposition 3.6.5. *We have*

$$\max_{r\in[\![0,4]\!]}\lambda^{2r}E_{\delta+1+r}\left(\nabla^r\mathfrak{h}\right)\lesssim e^{1+C(A_i)\varepsilon}\left(C(A_i)\varepsilon^4+\left(C_{\text{in}}^2+A_2^2\right)\varepsilon^2\right), \tag{3.6.49}$$

$$\max_{r\in[\![0,3]\!]}\lambda^{r+1}\left\|\partial_t^2\nabla^r\mathfrak{h}\right\|_{L^2_{\delta+r+2}}\lesssim e^{1+C(A_i)\varepsilon}\left(C(A_i)\varepsilon^2+\left(C_{\text{in}}+A_2\right)\varepsilon\right). \tag{3.6.50}$$

Proof. We start with the proof of (3.6.49). We commute ∇^r and $\tilde{\Box}_g$ (with $r\in[\![0,4]\!]$) to obtain the equation satisfied by $\nabla^r\mathfrak{h}$ from (3.4.8):

$$\tilde{\Box}_g\nabla^r\mathfrak{h}=\left[\tilde{\Box}_g,\nabla^r\right]\mathfrak{h}+\nabla^r\tilde{\Box}_g\mathfrak{h}. \tag{3.6.51}$$

From the energy estimate in Lemma 3.6.9 we need to estimate the $L^2_{\delta+1+r}$ norm of the RHS of (3.6.51). We estimate the commutator first. We have

$$\left\|\left[\tilde{\Box}_g,\nabla^r\right]\mathfrak{h}\right\|_{L^2_{\delta+1+r}}\lesssim\sum_{r_1+r_2=r-1}\left\|\nabla^{r_1+1}g\partial^2\nabla^{r_2}\mathfrak{h}\right\|_{L^2_{\delta+1+r}}$$

We have $r_1\leq 3$. If $r_1\leq 2$, then mimicking the proof of Lemma 3.6.1 gives $\left\|\nabla^{r_1+1}g\right\|_{L^\infty_{r_1}}\lesssim\frac{A_3\varepsilon}{\lambda^{r_1}}$ which in turn implies

$$\begin{aligned}\left\|\nabla^{r_1+1}g\partial^2\nabla^{r_2}\mathfrak{h}\right\|_{L^2_{\delta+1+r}}&\lesssim\left\|\nabla^{r_1+1}g\right\|_{L^\infty_{r_1}}\left\|\partial^2\nabla^{r_2}\mathfrak{h}\right\|_{L^2_{\delta+2+r_2}}\\&\lesssim\frac{A_3\varepsilon}{\lambda^{r_1}}\times\frac{A_3\varepsilon}{\lambda^{r_2+1}}\\&\lesssim C(A_i)\frac{\varepsilon^2}{\lambda^r}\end{aligned}$$

where we used (**BA3**). If $r_1=3$, then $r=4$ and $r_2=0$ we use the decomposition (3.6.2):

$$\left\|\nabla^4g\partial^2\mathfrak{h}\right\|_{L^2_{\delta+5}}\lesssim\left\|\nabla^4g_{\text{BG}}\partial^2\mathfrak{h}\right\|_{L^2_{\delta+5}}+\left\|\nabla^4\tilde{g}\partial^2\mathfrak{h}\right\|_{L^2_{\delta+5}}.$$

For the first term, we put ∇^4g_{BG} in L^∞_3 as in the proof of Lemma 3.6.1 and use (**BA3**). For the second term, we do as in the proof of Proposition 3.6.3, see (3.6.45) and (3.6.46). This proves that the commutator in (3.6.51) satisfies

$$\left\|\left[\tilde{\Box}_g,\nabla^r\right]\mathfrak{h}\right\|_{L^2_{\delta+1+r}}\lesssim C(A_i)\frac{\varepsilon^2}{\lambda^r}. \tag{3.6.52}$$

We now estimate the second term in (3.6.51). This has already been done in Lemma 3.6.5, we recall (3.6.27):

$$\left\|\nabla^r\tilde{\Box}_g\mathfrak{h}\right\|_{L^2_{\delta+r+1}}\lesssim\frac{C(A_i)\varepsilon^2+A_2\varepsilon}{\lambda^r}.$$

Together with (3.6.52) this implies that

$$\left\|\tilde{\Box}_g\nabla^r\mathfrak{h}\right\|_{L^2_{\delta+r+1}}\lesssim\frac{C(A_i)\varepsilon^2+A_2\varepsilon}{\lambda^r}. \tag{3.6.53}$$

We now apply Lemma 3.6.9 to (3.6.51) with $\sigma = \delta + r + 1$, this gives for $t \in [0,T]$ (after multiplication by λ^{2r})

$$\begin{aligned}
&\lambda^{2r} E_{\delta+r+1}(\nabla^r \mathfrak{h})(t) \\
&\quad \lesssim \lambda^{2r} E_{\delta+r+1}(\nabla^r \mathfrak{h})(0) \\
&\qquad + (1 + C(A_i)\varepsilon) \int_0^t \lambda^{2r} E_{\delta+r+1}(\nabla^r \mathfrak{h})(s) \mathrm{d}s + \int_0^t \lambda^{2r} \left\| \tilde{\Box}_g \nabla^r \mathfrak{h} \right\|^2_{L^2_{\delta+r+1}} (s) \mathrm{d}s \\
&\quad \lesssim \lambda^{2r} E_{\delta+r+1}(\nabla^r \mathfrak{h})(0) + C(A_i)\varepsilon^4 + A_2^2 \varepsilon^2 \\
&\qquad + (1 + C(A_i)\varepsilon) \int_0^t \lambda^{2r} E_{\delta+r+1}(\nabla^r \mathfrak{h})(s) \mathrm{d}s
\end{aligned}$$

We apply Gronwall's lemma to obtain

$$\lambda^{2r} E_{\delta+r+1}(\nabla^r \mathfrak{h})(t) \lesssim e^{1+C(A_i)\varepsilon} \left(\lambda^{2r} E_{\delta+r+1}(\nabla^r \mathfrak{h})(0) + C(A_i)\varepsilon^4 + A_2^2 \varepsilon^2 \right).$$

In order to estimate $E_{\delta+r+1}(\nabla^r \mathfrak{h})(0)$, we use (3.4.44):

$$\lambda^{2r} E_{\delta+r+1}(\nabla^r \mathfrak{h})(0) \leq C_{\mathrm{in}}^2 \varepsilon^2.$$

This concludes the proof of (3.6.49).

We now turn to the proof of (3.6.50). For this we write the operator $\tilde{\Box}_g$ in the following schematic way:

$$\tilde{\Box}_g = g^{tt} \partial_t^2 + g \partial \nabla.$$

Thanks to the background regularity we can divide by g^{tt} and therefore obtain for $r \leq 3$:

$$\begin{aligned}
\left\| \partial_t^2 \nabla^r \mathfrak{h} \right\|_{L^2_{\delta+r+2}} &\lesssim \sum_{r_1+r_2=r} \left\| \nabla^{r_1} g \nabla^{r_2} \tilde{\Box}_g \mathfrak{h} \right\|_{L^2_{\delta+r+2}} + \sum_{r_1+r_2+r_3=r} \left\| \nabla^{r_1} g \nabla^{r_2} g \partial \nabla^{r_3+1} \mathfrak{h} \right\|_{L^2_{\delta+r+2}} \\
&=: I + II.
\end{aligned}$$

For I, we use (3.6.5), the background regularity and (3.6.27):

$$I \lesssim \frac{C(A_i)\varepsilon^2 + A_2 \varepsilon}{\lambda^r}.$$

For II we proceed similarly but use (3.6.49) instead:

$$II \lesssim \frac{e^{1+C(A_i)\varepsilon} \left(C(A_i)\varepsilon^2 + (C_{\mathrm{in}} + A_2)\varepsilon \right)}{\lambda^{r+1}}.$$

This concludes the proof. □

3.6.6 Conclusion of the bootstrap

Looking at Propositions 3.6.1-3.6.4-3.6.5, we see that a choice of constants A_i such that $A_1 \ll A_2 \ll A_3$ together with ε and λ small enough compared to 1 (but with ε still independent from λ) allows us to improve the bootstrap assumptions (**BA1**)-(**BA2**)-(**BA3**), as long as their time of existence of $(\mathfrak{F}, \mathfrak{h})$ is less than 1. This shows that the time of existence of the solution $(\mathfrak{F}, \mathfrak{h})$ is actually equal to 1. We have proved the following theorem.

Theorem 3.6.1. *Given initial data as in Corollary 3.4.1 and if ε and λ are small enough (but with ε still independent from λ), there exists a solution $(\mathfrak{F},\mathfrak{h})$ to the reduced system (3.4.7)-(3.4.8) on $[0,1]\times\mathbb{R}^3$. Moreover there exists a numerical constant $C>0$ such that (**BA1**)-(**BA2**)-(**BA3**) hold with the A_i replaced by C.*

The estimates that $(\mathfrak{F},\mathfrak{h})$ satisfied, along with the ones satisfied by $F^{(1)}$, $F^{(2,1)}$ and $F^{(2,2)}$ (see Theorem 3.5.1) prove the main estimate (3.1.25) in Theorem 3.1.2 if one define

$$\tilde{\mathfrak{h}}_\lambda = \mathfrak{h} + \lambda g^{(3)}.$$

It only remains to prove that g given by (3.3.1) solves the Einstein vacuum equations.

3.7 Solving the Einstein vacuum equations

By proving Theorems 3.5.1 and 3.6.1, we construct a metric g given by (3.3.1) solution to both the background and the reduced system on the manifold $[0,1]\times\mathbb{R}^3$. In this section, we conclude the proof of Theorem 3.1.2 by proving that g is actually a solution to the Einstein vacuum equations, i.e that $R_{\mu\nu}(g)=0$ on the same manifold. We will show in Section 3.7.1 that $R_{\mu\nu}(g)$ contains only polarization condition tensors (see Section 3.4.4.2) or the generalised wave gauge term Υ^ρ. The fact that these terms vanish will be proved in Sections 3.7.3 and 3.7.4 using the contracted Bianchi identities, i.e the fact that the Einstein tensor of g is divergence free.

3.7.1 The reduced Ricci tensor

In this section we compute the Ricci tensor of the solution g to the background and reduced systems. We show that it contains only $V^{(2,i)}$ or Υ terms, which explains afterwards the exact expression of the equations (3.4.4)-(3.4.8). This is based on the high-frequency expansion of the Ricci tensor performed in Section 3.3.2, whose notations we follow. The results of this section imply that the Ricci tensor of g formally admits the following expansion

$$R_{\mu\nu} = R^{(0)}_{\mu\nu} + \lambda R^{(1)}_{\mu\nu} + \lambda^2 R^{(\geq 2)}_{\mu\nu}. \tag{3.7.1}$$

Before we start computing the terms in (3.7.1), let us obtain the polarization of $\mathfrak{F}$. Thanks to the RHS of (3.4.7) having a polarization tensor equal to 0 and thanks to (3.4.29) and (3.4.47), it is easy to mimic the proof of Lemma 3.5.1 and to obtain the following:

Lemma 3.7.1. *We have* $\mathrm{Pol}(\mathfrak{F})=0$ *and* $\mathfrak{F}_{L_0\underline{L}_0}=0$ *on* $[0,1]\times\mathbb{R}^3$.

Recalling (3.3.10) and (3.3.17), this lemma implies in particular that the expression of $W^{(1)}_{\alpha\beta}$ and H^ρ simplify and become

$$W^{(1)}_{\alpha\beta} = \cos\left(\frac{u_0}{\lambda}\right) W^{(1,1)}_{\alpha\beta} + \sin\left(\frac{2u_0}{\lambda}\right) W^{(1,2)}_{\alpha\beta} + \cos\left(\frac{u_0}{\lambda}\right)\cos\left(\frac{2u_0}{\lambda}\right) F^{(2,2)}_{L_0L_0}F^{(1)}_{\alpha\beta},$$
$$H^\rho = \lambda (H^{(1)})^\rho + \lambda^2\left((H^{(2)})^\rho + \Upsilon^\rho\right) + \lambda^3 (H^{(\geq 3)})^\rho.$$

3.7.1.1 The zeroth order

We start by computing the zeroth order of $R_{\mu\nu}(g)$, denoted $R^{(0)}_{\mu\nu}$ and which satisfies

$$2R^{(0)}_{\alpha\beta} = -W^{(0)}_{\alpha\beta} + (g_0)_{\rho(\alpha}\partial_{\beta)}u_0\partial_\theta (H^{(1)})^\rho + P^{(0)}_{\alpha\beta}. \tag{3.7.2}$$

Lemma 3.7.2. *Given $F^{(1)}$ solution of (3.4.4) we have*

$$R^{(0)}_{\alpha\beta} = -\frac{1}{2}\sin\left(\frac{u_0}{\lambda}\right)\partial_{(\alpha}u_0V^{(2,1)}_{\beta)} - 2\cos\left(\frac{2u_0}{\lambda}\right)\partial_{(\alpha}u_0V^{(2,2)}_{\beta)}.$$

Proof. We use (3.3.6), (3.3.19) and (3.3.28) to rewrite (3.7.2). We note that the Christoffel symbols in $P^{(0)}_{\alpha\beta}\left[g^{(1)}\right]$ combined with the transport operator in $W^{(0)}_{\alpha\beta}$ actually give the operator $\mathscr{L}_0$ since in coordinates we have

$$\mathbf{D}_{L_0}T_{\alpha\beta} = L_0T_{\alpha\beta} - L_0^\rho\Gamma(g_0)^\nu_{(\alpha\rho}T_{\beta)\nu}$$

for any symmetric 2-tensor T. Moreover we use (3.5.16). We obtain

$$\begin{aligned}2R^{(0)}_{\alpha\beta} &= -\tilde{\Box}_{g_0}(g_0)_{\alpha\beta} + P_{\alpha\beta}(g_0)(\partial g_0,\partial g_0) - 2F_0^2\partial_\alpha u_0\partial_\beta u_0 + \sin\left(\frac{u_0}{\lambda}\right)\mathscr{L}_0F^{(1)}_{\alpha\beta}\\ &\quad - \sin\left(\frac{u_0}{\lambda}\right)\partial_{(\alpha}u_0\left(\mathrm{Pol}_{\beta)}\left(F^{(2,1)}\right) + (g_0)_{\rho\beta)}(\tilde{H}^{(1,1)})^\rho + \hat{P}^{(0,1)}_{\beta)}\left[g^{(1)}\right]\right)\\ &\quad + 2\cos\left(\frac{2u_0}{\lambda}\right)\partial_{(\alpha}u_0\left(-2\mathrm{Pol}_{\beta)}\left(F^{(2,2)}\right) + (g_0)_{\rho\beta)}(\tilde{H}^{(1,2)})^\rho + \frac{1}{2}\hat{P}^{(0,2)}_{\beta)}\left[g^{(1)}\right]\right).\end{aligned}$$

Thanks to (3.4.4) and the equation (3.1.7) satisfied by the background metric g_0, the fist line of this expression vanishes. Moreover, the second and third line can be rewritten with $V^{(2,1)}$ and $V^{(2,2)}$, see (3.4.12) and (3.4.13). This concludes the proof. □

3.7.1.2 The first order

In the following computations of $R^{(1)}_{\alpha\beta}$ we will encounter a lot of terms proportional to the polarization terms $V^{(2,j)}$ for which a precise expression is useless. Therefore, we introduce the following practical notation: we denote by $O\left(V^{(2)}\right)$ any combination of the tensors $V^{(2,j)}$ or its derivatives for $j=1,2$. This notation extends to tensors, in the sense that $O^\alpha\left(V^{(2)}\right)$ will denote any 1-tensor with coefficients of the form $O\left(V^{(2)}\right)$.

Let us compute $R^{(1)}_{\alpha\beta}$. Since $(\partial g)^{(0)} = \partial g_0 - \sin\left(\frac{u_0}{\lambda}\right)\partial u_0 F^{(1)}$ we have

$$\begin{aligned}2R^{(1)}_{\alpha\beta} &= -W^{(1)}_{\alpha\beta} + (H^{(1)})^\rho\left(\partial_\rho(g_0)_{\alpha\beta} - \sin\left(\frac{u_0}{\lambda}\right)\partial_\rho u_0F^{(1)}_{\alpha\beta}\right) + (g_0)_{\rho(\alpha}\partial_{\beta)}(H^{(1)})^\rho \qquad (3.7.3)\\ &\quad + (g_0)_{\rho(\alpha}\partial_{\beta)}u_0\partial_\theta(H^{(2)})^\rho + \cos\left(\frac{u_0}{\lambda}\right)F^{(1)}_{\rho(\alpha}\partial_{\beta)}u_0\partial_\theta(H^{(1)})^\rho + P^{(1)}_{\alpha\beta}\\ &\quad + 2R^{(1)}_{\alpha\beta}(\Upsilon) \qquad (3.7.4)\end{aligned}$$

where by $\partial_{\beta)}(H^{(1)})^\rho$ we denote the derivative of $(H^{(1)})^\rho$ without differenciating its oscillating parts and where $2R^{(1)}_{\alpha\beta}(\Upsilon)$ contains all the contributions of Υ^ρ to the λ^1 level of the Ricci tensor. We don't give the exact form of these contributions yet.

Lemma 3.7.3. *Given $F^{(2,1)}$, $F^{(2,2)}$, $\mathfrak{F}$ and $g^{(3)}$ solutions to (3.4.5), (3.4.6), (3.4.7) and (3.4.18) we have*

$$\begin{aligned}R^{(1)}_{\alpha\beta} &= \frac{1}{2}\cos\left(\frac{u_0}{\lambda}\right)\mathbf{D}_{(\alpha}V^{(2,1)}_{\beta)} - \sin\left(\frac{2u_0}{\lambda}\right)\mathbf{D}_{(\alpha}V^{(2,2)}_{\beta)} + O\left(V^{(2)}\right)F^{(1)}_{\alpha\beta}\\ &\quad - \frac{1}{2}\cos\left(\frac{u_0}{\lambda}\right)\Pi_{\geq}\left(\mathfrak{h}_{L_0L_0}\right)F^{(1)}_{\alpha\beta} + R^{(1)}_{\alpha\beta}(\Upsilon).\end{aligned}$$

Proof. We decompose $W^{(1)}_{\alpha\beta}$ using first (3.3.10) and then (3.3.11) and (3.3.12), and we decompose $P^{(1)}_{\alpha\beta}$ using first (3.3.33) together with (3.3.34), (3.3.38) and (3.3.35). Again we note that the Christoffel symbols in (3.3.35) with the transport operator in (3.3.11) and (3.3.12) form $\mathscr{L}_0 F^{(2,i)}$. We obtain

$$\begin{aligned}
-W^{(1)}_{\alpha\beta} + P^{(1)}_{\alpha\beta} = &- \cos\left(\frac{u_0}{\lambda}\right)\left(\mathscr{L}_0 F^{(2,1)}_{\alpha\beta} + \tilde{W}^{(1,1)}_{\alpha\beta} - P^{(1,1)}_{\alpha\beta}\right) \\
&- \cos\left(\frac{u_0}{\lambda}\right)\left(\mathscr{L}_0 \mathfrak{F}_{\alpha\beta} + \mathfrak{h}_{L_0L_0} F^{(1)}_{\alpha\beta}\right) \\
&+ 2\sin\left(\frac{2u_0}{\lambda}\right)\left(\mathscr{L}_0 F^{(2,2)}_{\alpha\beta} + \tilde{W}^{(1,2)}_{\alpha\beta} + \frac{1}{2}P^{(1,2)}_{\alpha\beta} - \frac{1}{4}F^{(2,1)}_{L_0L_0}F^{(1)}_{\alpha\beta}\right) \\
&+ \partial_{(\alpha}u_0 \hat{P}^{(1)}_{\beta)}\left[g^{(2)}\right] + \cos\left(\frac{3u_0}{\lambda}\right)\partial_{(\alpha}u_0 \hat{P}^{(1,3)}_{\beta)} \\
&- \cos\left(\frac{u_0}{\lambda}\right)\cos\left(\frac{2u_0}{\lambda}\right) F^{(2,2)}_{L_0L_0}F^{(1)}_{\alpha\beta}
\end{aligned} \tag{3.7.5}$$

In this expression, the last two lines include forbidden frequencies at the λ level, the first one will be absorbed by the polarization of $g^{(3)}$ and we can rewrite the second one in terms of polarization conditions, thanks to (3.4.17). Therefore the last line in (3.7.5) is of the form $O\left(V^{(2)}\right)F^{(1)}_{\alpha\beta}$.

Let us now look at the terms in (3.7.3) depending on $H^{(1)}$ or $H^{(2)}$. For $H^{(2)}$, we use (3.3.22) and for $H^{(1)}$ we rewrite it using $V^{(2,i)}$ for $i=1,2$ (see (3.4.12) and (3.4.13)):

$$\begin{aligned}
(H^{(1)})^\rho = &- \cos\left(\frac{u_0}{\lambda}\right) g_0^{\rho\sigma}\hat{P}^{(0,1)}_\sigma\left[g^{(1)}\right] - \frac{1}{2}\sin\left(\frac{2u_0}{\lambda}\right) g_0^{\rho\sigma}\hat{P}^{(0,2)}_\sigma\left[g^{(1)}\right] \\
&+ \cos\left(\frac{u_0}{\lambda}\right) g_0^{\rho\sigma}V^{(2,1)}_\sigma - 2\sin\left(\frac{2u_0}{\lambda}\right) g_0^{\rho\sigma}V^{(2,2)}_\sigma.
\end{aligned} \tag{3.7.6}$$

We can split the terms in (3.7.3) depending on $H^{(1)}$ or $H^{(2)}$ into three categories:

- admissible frequencies, i.e θ and 2θ, which only depends on $g^{(1)}$,
- terms depending on $g^{(\leq 2)}$ with both admissible and forbidden frequencies but with the non-tangential structure,
- terms depending on $V^{(2,i)}$ and $\mathbf{D}V^{(2,i)}$, the former are of the form $O\left(V^{(2)}\right)$ but we keep the latter as they are.

More precisely, using (3.7.6) and (3.3.22) we obtain

$$\begin{aligned}
&2R^{(1)}_{\alpha\beta} + W^{(1)}_{\alpha\beta} - P^{(1)}_{\alpha\beta} - 2R^{(1)}_{\alpha\beta}(\Upsilon) \\
&= -\cos\left(\frac{u_0}{\lambda}\right)\mathbf{D}_{(\alpha}\hat{P}^{(0,1)}_{\beta)}\left[g^{(1)}\right] - \frac{1}{2}\sin\left(\frac{2u_0}{\lambda}\right)\left(\mathbf{D}_{(\alpha}\hat{P}^{(0,2)}_{\beta)}\left[g^{(1)}\right] + L_0^\sigma\hat{P}^{(0,1)}_\sigma\left[g^{(1)}\right]F^{(1)}_{\alpha\beta}\right) \qquad (3.7.7) \\
&\quad + \cos\left(\frac{u_0}{\lambda}\right)\mathbf{D}_{(\alpha}V^{(2,1)}_{\beta)} - 2\sin\left(\frac{2u_0}{\lambda}\right)\mathbf{D}_{(\alpha}V^{(2,2)}_{\beta)} + O\left(V^{(2)}\right)F^{(1)}_{\alpha\beta} \\
&\quad + \partial_{(\alpha}u_0 \mathrm{Pol}_{\beta)}\left(\partial^2_\theta g^{(3)}\right) + \partial_{(\alpha}u_0 (g_0)_{\rho\beta)}\partial_\theta(\tilde{H}^{(2)})^\rho + \cos\left(\frac{u_0}{\lambda}\right)\partial_{(\alpha}u_0 F^{(1)}_{\rho\beta)}\partial_\theta(H^{(1)})^\rho \\
&\quad - \frac{1}{2}\sin\left(\frac{u_0}{\lambda}\right)\sin\left(\frac{2u_0}{\lambda}\right)L_0^\sigma\hat{P}^{(0,2)}_\sigma\left[g^{(1)}\right]F^{(1)}_{\alpha\beta}
\end{aligned}$$

where we also used the following simple fact true for all Ω 1-tensor:

$$g_0^{\rho\sigma}\Omega_\sigma\partial_\rho(g_0)_{\alpha\beta} + (g_0)_{\rho(\alpha}\partial_{\beta)}\left(g_0^{\rho\sigma}\Omega_\sigma\right) = \mathbf{D}_{(\alpha}\Omega_{\beta)}. \tag{3.7.8}$$

In this expression, note that the last line corresponds to a forbidden frequency without the non-tangential structure, but thanks to (3.3.32) this line vanishes. This is due to the weak polarized null condition. With this in mind, we add (3.7.5) and (3.7.7) and use (3.4.5), (3.4.6) and (3.4.7) to cancel the admissible frequencies:

$$\begin{aligned}
&2R^{(1)}_{\alpha\beta} - 2R^{(1)}_{\alpha\beta}(\Upsilon) \\
&= -\cos\left(\frac{u_0}{\lambda}\right)\Pi_{\geq}\left(\mathfrak{h}_{L_0L_0}\right)F^{(1)}_{\alpha\beta} \\
&\quad + \cos\left(\frac{u_0}{\lambda}\right)\mathbf{D}_{(\alpha}V^{(2,1)}_{\beta)} - 2\sin\left(\frac{2u_0}{\lambda}\right)\mathbf{D}_{(\alpha}V^{(2,2)}_{\beta)} + O\left(V^{(2)}\right)F^{(1)}_{\alpha\beta} \\
&\quad + \partial_{(\alpha}u_0\mathrm{Pol}_{\beta)}\left(\partial_\theta^2 g^{(3)}\right) + \partial_{(\alpha}u_0(g_0)_{\rho\beta)}\partial_\theta(\tilde{H}^{(2)})^\rho + \cos\left(\frac{u_0}{\lambda}\right)\partial_{(\alpha}u_0F^{(1)}_{\rho\beta)}\partial_\theta(H^{(1)})^\rho \\
&\quad + \partial_{(\alpha}u_0\hat{P}^{(1)}_{\beta)}\left[g^{(2)}\right] + \cos\left(\frac{3u_0}{\lambda}\right)\partial_{(\alpha}u_0\hat{P}^{(1,3)}_{\beta)}
\end{aligned}$$

We conclude the proof using (3.4.18), which shows how $g^{(3)}$ absorbs the terms with the non-tangential structure. □

3.7.1.3 The higher orders

Since the higher levels of the Ricci tensor are canceled via the wave equation for the remainder $\mathfrak{h}$, we don't need a very precise expression of $R^{(\geq 2)}_{\alpha\beta}$ and we simply write

$$2R^{(\geq 2)}_{\alpha\beta} = -W^{(\geq 2)}_{\alpha\beta} + \left(\mathring{H}^\rho\partial_\rho g_{\alpha\beta} + g_{\rho(\alpha}\partial_{\beta)}\mathring{H}^\rho\right)^{(\geq 2)} + 2R^{(\geq 2)}_{\alpha\beta}(\Upsilon) + P^{(\geq 2)}_{\alpha\beta}$$

where we recall that $\mathring{H}^\rho = H^\rho - \lambda^2\Upsilon^\rho$ and where $R^{(\geq 2)}_{\alpha\beta)}(\Upsilon)$ contains all the contributions from Υ^ρ at the order λ^2 or higher in the Ricci tensor. Note that

$$\lambda R^{(1)}_{\alpha\beta}(\Upsilon) + \lambda^2 R^{(\geq 2)}_{\alpha\beta}(\Upsilon) = \frac{\lambda^2}{2}\left(\Upsilon^\rho\partial_\rho g_{\alpha\beta} + g_{\rho(\alpha}\partial_{\beta)}\Upsilon^\rho\right) \tag{3.7.9}$$

where on the RHS of (3.7.9) the derivatives in $\partial_\rho g_{\alpha\beta}$ and $\partial_{\beta)}\Upsilon^\rho$ also hit the oscillating parts (producing $R^{(1)}_{\alpha\beta}(\Upsilon)$).

Lemma 3.7.4. *Given $\mathfrak{h}$ solution to (3.4.8) we have*

$$R^{(\geq 2)}_{\alpha\beta} = \frac{1}{2\lambda}\cos\left(\frac{u_0}{\lambda}\right)\Pi_{\geq}\left(\mathfrak{h}_{L_0L_0}\right)F^{(1)}_{\alpha\beta} + R^{(\geq 2)}_{\alpha\beta}(\Upsilon)$$

The term proportional to $\frac{1}{\lambda}\Pi_{\geq}\left(\mathfrak{h}_{L_0L_0}\right)$ in $R^{(\geq 2)}$ cancels out with the same term present in $R^{(1)}$ so that it disappears from the final Einstein tensor computed in the next sections.

3.7.2 The Einstein tensor and the contracted Bianchi identities

As Lemmas 3.7.2-3.7.3-3.7.4 shows, the Ricci tensor of g given by (3.3.1) contains only gauge terms or the polarization conditions, i.e terms depending only on Υ and $V^{(2,i)}$. In order to show that g is solution of the Einstein vacuum equations, it thus remains to show that they vanish.

This is the content of Sections 3.7.3 and 3.7.4 and it is proved using the contracted Bianchi identities, i.e the fact that the Einstein tensor of any metric is divergence free, which will give us extra equations satisfied by $V^{(2,i)}$ and Υ. Even though this is standard when working with the (generalised or not) wave coordinates, the high-frequency character of g (which manifests at this point with the presence of the polarization conditions tensors $V^{(2,i)}$) changes the situation. Schematically, we will show that the divergence of G satisfies

$$\mathrm{div}_g G = \mathrm{T}\left(\frac{u_0}{\lambda}\right)\mathscr{L}_0 V^{(2,i)} + \lambda^2\tilde{\Box}_g\Upsilon$$

with T a trigonometric function (see (3.2.4)). The contracted Bianchi identities precisely state that for all $\lambda\in(0,\lambda_0]$ we have

$$\mathrm{T}\left(\frac{u_0}{\lambda}\right)\mathscr{L}_0 V^{(2,i)} + \lambda^2\tilde{\Box}_g\Upsilon = 0 \tag{3.7.10}$$

with $\mathscr{L}_0 V^{(2,i)}$ independant of λ and $\tilde{\Box}_g\Upsilon$ depending on λ through g and Υ. Our goal is to extract from (3.7.10) the two equations

$$\mathscr{L}_0 V^{(2,i)} = 0 \quad \text{and} \quad \tilde{\Box}_g\Upsilon = 0.$$

However, because of the oscillating term in (3.7.10) and the fact that $\tilde{\Box}_g\Upsilon$ depends on λ (since Υ depends on λ through $\mathfrak{h}$ or $\mathfrak{F}$), we can't simply consider the expression in (3.7.10) as a polynomial in λ vanishing on the interval $(0,\lambda_0]$. Instead we multiply (3.7.10) by $\mathrm{T}\left(\frac{u_0}{\lambda}\right)$ and consider the weak limit in L^2 when λ tends to 0 (see the following remark). In order to consider this limit for $\tilde{\Box}_g\Upsilon$ we need precise estimates obtained in Lemma 3.7.6 and which follows from our bootstrap assumptions for $\mathfrak{h}$ and $\mathfrak{F}$. This strategy allows us to show that $V^{(2,i)} = \Upsilon = 0$ on the whole spacetime, see Propositions 3.7.1 and 3.7.2.

Remark 3.7.1. *Let* T *be a trigonometric function, i.e an element of* (3.2.4). *Let us be more precise on the convergence of the family* $\left(\mathrm{T}\left(\frac{u_0}{\lambda}\right)\right)_{\lambda\in(0,\lambda_0]}$ *when* λ *tends to 0. This convergence will be crucially used in Lemma 3.7.9 and Proposition 3.7.1 for terms of the form* $\mathrm{T}\left(\frac{u_0}{\lambda}\right) f$ *with* f *compactly supported and* $\mathcal{C}^1$*, and therefore we only care about the convergence of* $\left(\mathrm{T}\left(\frac{u_0}{\lambda}\right)\right)_{\lambda\in(0,\lambda_0]}$ *in* $L^2(K)$ *for* K *a compact of* $\mathbb{R}^3$*. Let* ψ *a test function supported in* K*. Thanks to* (3.1.10)*, we can use a stationnary phase argument and prove*

$$\left|\int_K \mathrm{T}\left(\frac{u_0}{\lambda}\right)\psi\right| \lesssim \lambda\left\|\psi\right\|_{W^{1,\infty}(K)}. \tag{3.7.11}$$

Using now the density of test functions in $L^2(K)$*, we deduce from* (3.7.11) *that*

$$\int_K \mathrm{T}\left(\frac{u_0}{\lambda}\right)\psi$$

tends to 0 as λ *tends to 0 for all* $\psi\in L^2(K)$*. This shows that the family* $\left(\mathrm{T}\left(\frac{u_0}{\lambda}\right)\right)_{\lambda\in(0,\lambda_0]}$ *converges weakly to 0 in* $L^2(K)$*. In the sequel, the compact* K *will be*

$$K := \{|x| \leq C_{\mathrm{supp}}R\},$$

i.e the maximal support of all the background perturbations, and in particular the support of $V^{(2,i)}$*.*

3.7.2.1 The Einstein tensor

For g the metric solution of the background and reduced systems on $[0,1]\times\mathbb{R}^3$ we define its Einstein tensor by

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} \tag{3.7.12}$$

where $R = g^{\mu\nu}R_{\mu\nu}$ is the scalar curvature of g. The terms $V^{(2,i)}$ and Υ will be treated differently so it is useful to decompose G in the following way:

$$G_{\alpha\beta} = G\left(V\right)_{\alpha\beta} + G\left(\Upsilon\right)_{\alpha\beta}. \tag{3.7.13}$$

with

$$G\left(\Upsilon\right)_{\alpha\beta} = \frac{\lambda^2}{2}\left(g_{\rho(\alpha}\partial_{\beta)}\Upsilon^\rho - g_{\alpha\beta}\partial_\rho\Upsilon^\rho + \Upsilon^\rho\partial_\rho g_{\alpha\beta} - \frac{1}{2}g^{\mu\nu}\Upsilon^\rho\partial_\rho g_{\mu\nu}g_{\alpha\beta}\right) \tag{3.7.14}$$

and $G(V)$ depending only on $V^{(2,i)}$. Thanks to Lemmas 3.7.2-3.7.3-3.7.4, the tensor $G(V)$ admits a high-frequency expansion

$$G\left(V\right)_{\alpha\beta} = G^{(0)}_{\alpha\beta} + \lambda G^{(1)}_{\alpha\beta} + O\left(\lambda^2\right).$$

The following lemma gives the expression of $G^{(0)}_{\alpha\beta}$ and $G^{(1)}_{\alpha\beta}$, we leave the proof to the reader as it is a direct application of Lemmas 3.7.2-3.7.3 (which in particular allow us to compute the high-frequency expansion of the scalar curvature).

Lemma 3.7.5. *We have*

$$\begin{aligned} G^{(0)}_{\alpha\beta} &= -\frac{1}{2}\sin\left(\frac{u_0}{\lambda}\right)\left(\partial_{(\alpha}u_0V^{(2,1)}_{\beta)} + V^{(2,1)}_{L_0}(g_0)_{\alpha\beta}\right) \\ &\quad - 2\cos\left(\frac{2u_0}{\lambda}\right)\left(\partial_{(\alpha}u_0V^{(2,2)}_{\beta)} + V^{(2,2)}_{L_0}(g_0)_{\alpha\beta}\right), \end{aligned} \tag{3.7.15}$$

$$\begin{aligned} G^{(1)}_{\alpha\beta} &= \frac{1}{2}\cos\left(\frac{u_0}{\lambda}\right)\left(\mathbf{D}_{(\alpha}V^{(2,1)}_{\beta)} - \mathrm{div}_{g_0}V^{(2,1)}(g_0)_{\alpha\beta}\right) \\ &\quad - \sin\left(\frac{2u_0}{\lambda}\right)\left(\mathbf{D}_{(\alpha}V^{(2,2)}_{\beta)} - \mathrm{div}_{g_0}V^{(2,2)}(g_0)_{\alpha\beta}\right) + O\left(V^{(2)}\right)F^{(1)}_{\alpha\beta}. \end{aligned} \tag{3.7.16}$$

3.7.2.2 The contracted Bianchi identities

The (contracted) Bianchi identities precisely reads

$$\mathrm{div}_g G = 0. \tag{3.7.17}$$

From the decomposition (3.7.13) it suffices to compute $\mathrm{div}_g G(\Upsilon)$ and $\mathrm{div}_g G(V)$. The former is computed and studied in Section 3.7.2.3 while the latter formally admits the following high-frequency expansion:

$$\mathrm{div}_g G\left(V\right)_\alpha = \frac{1}{\lambda}\mathcal{B}(V)^{(-1)}_\alpha + \mathcal{B}(V)^{(0)}_\alpha + O\left(\lambda\right) \tag{3.7.18}$$

The terms $\mathcal{B}(V)^{(i)}_\alpha$ for $i=-1,0$ will be computed in Section 3.7.3.

3.7.2.3 Divergence of $G(\Upsilon)$

In this section, we compute and study the divergence of the gauge terms associated to the wave equation, i.e $G(\Upsilon)$, whose expression is given by (3.7.14). The expression in coordinates

$$\mathrm{div}_g G(\Upsilon)_\alpha = g^{\mu\nu}\partial_\mu G(\Upsilon)_{\nu\alpha} - g^{\mu\nu}\Gamma^\rho_{\mu(\alpha}G(\Upsilon)_{\rho\nu)}$$

and a standard computation first gives

$$\mathrm{div}_g G(\Upsilon)_\alpha = \frac{\lambda^2}{2}\left(g_{\rho\alpha}\tilde{\Box}_g\Upsilon^\rho + \tilde{\mathcal{B}}_\alpha\right) \tag{3.7.19}$$

with

$$\begin{aligned}\tilde{\mathcal{B}}_\alpha &= g^{\mu\nu}\partial_{(\alpha}\Upsilon^\rho\partial_\mu g_{\rho\nu)} - g^{\mu\nu}\partial_\rho\Upsilon^\rho\partial_\mu g_{\nu\alpha} \\ &\quad + g^{\mu\nu}\partial_\mu\left(\Upsilon^\rho\partial_\rho g_{\alpha\nu} - \frac{1}{2}g^{\rho\sigma}\Upsilon^\gamma\partial_\gamma g_{\rho\sigma}g_{\alpha\nu}\right) - g^{\mu\nu}\Gamma^\rho_{\mu(\alpha}G(\Upsilon)_{\rho\nu)}.\end{aligned}$$

Recall the definition of Υ given in (3.3.16). We study $\mathrm{div}_g G(\Upsilon)$ in the following lemmas where we are careful about the amount of derivatives involved and the orders in λ. In the proof, we only consider the $\partial\mathfrak{h}$ and $\partial\mathfrak{F}$ terms in Υ^ρ (i.e the first line in (3.3.16)) since they satisfy the worst estimates. Moreover, since we only care about $L^2(K)$ norms with K compact we can neglect the weights for $\mathfrak{h}$.

Lemma 3.7.6. *We have*

$$\left\|\mathrm{div}_g G(\Upsilon)_\alpha\right\|_{L^2(K)} \lesssim \lambda.$$

Proof. In this proof, we use the notation $L^2 = L^2(K)$ for clarity. We start with $\tilde{\mathcal{B}}_\alpha$. Recall that g and ∂g are bounded in L^∞ irrespective of λ. Moreover, $\partial\mathfrak{h}$ and $\partial\mathfrak{F}$ are bounded in L^2 irrespective of λ thanks to (**BA1**) and (**BA3**). Therefore, the only terms potentially losing one power of λ in $\tilde{\mathcal{B}}_\alpha$ are $\partial\Upsilon$ and $\partial^2 g$. Indeed thanks to (3.3.1), (**BA1**) and (**BA3**) those terms are bounded by $\frac{1}{\lambda}$ in L^2, which shows that

$$\left\|\tilde{\mathcal{B}}_\alpha\right\|_{L^2} \lesssim \frac{1}{\lambda}. \tag{3.7.20}$$

Let us now look at $\tilde{\Box}_g\Upsilon^\rho(\mathfrak{h})$. Again, we only focus on the main terms in $\Upsilon^\rho(\mathfrak{h})$, that is $\partial\mathfrak{h} + \sin\left(\frac{u_0}{\lambda}\right)\partial\mathfrak{F}$. For $\partial\mathfrak{h}$, we use the equation (3.4.8) and commute it with one derivative to obtain

$$\tilde{\Box}_g\partial\mathfrak{h} = \partial\tilde{\Box}_g\mathfrak{h} + [\tilde{\Box}_g, \partial]\mathfrak{h}.$$

We have

$$\left\|[\tilde{\Box}_g, \partial]\mathfrak{h}\right\|_{L^2} \lesssim \|\partial g\|_{L^\infty}\left\|\partial^2\mathfrak{h}\right\|_{L^2} \lesssim \frac{1}{\lambda}$$

where we used (**BA3**). We now use the equation (3.4.8) (again, we only write down the terms depending on $\mathfrak{F}$ with the highest derivative):

$$\left\|\partial\tilde{\Box}_g\mathfrak{h}\right\|_{L^2} \lesssim \frac{1}{\lambda}\left\|\tilde{\Box}_g\mathfrak{F}\right\|_{L^2} + \left\|\nabla\tilde{\Box}_g\mathfrak{F}\right\|_{L^2} + \left\|\partial_t\tilde{\Box}_g\mathfrak{F}\right\|_{L^2} \lesssim \frac{1}{\lambda} + \left\|\partial_t\tilde{\Box}_g\mathfrak{F}\right\|_{L^2}$$

where we used (**BA2**). This estimate can't directly be used to bound $\partial_t\tilde{\Box}_g\mathfrak{F}$, instead we use (3.6.17):

$$\begin{aligned}\left\|\partial_t\tilde{\Box}_g\mathfrak{F}\right\|_{L^2} &\lesssim \left\|[\partial_t,\tilde{\Box}_g]\mathfrak{F}\right\|_{L^2} + \left\|\tilde{\Box}_g\partial_t\mathfrak{F}\right\|_{L^2}\\ &\lesssim \left\|\partial^2\mathfrak{F}\right\|_{L^2} + \left\|\nabla\tilde{\Box}_g\mathfrak{F}\right\|_{L^2} + \left\|\partial^2\mathfrak{h}\right\|_{L^2}\\ &\lesssim \frac{1}{\lambda}.\end{aligned}$$

We proved that

$$\left\|\tilde{\Box}_g\partial\mathfrak{h}\right\|_{L^2} \lesssim \frac{1}{\lambda}. \tag{3.7.21}$$

For the $\sin\left(\frac{u_0}{\lambda}\right)\partial\mathfrak{F}$ term in $\Upsilon^\rho(\mathfrak{h})$ we use (3.2.5) to deduce that we need to estimate

$$\frac{1}{\lambda}\cos\left(\frac{u_0}{\lambda}\right)\mathscr{L}_0\partial\mathfrak{F} + \sin\left(\frac{u_0}{\lambda}\right)\left(\tilde{\Box}_g\partial\mathfrak{F} - \frac{1}{\lambda^2}g^{-1}(\mathrm{d}u_0,\mathrm{d}u_0)\partial\mathfrak{F}\right).$$

First, using (3.3.7), (**BA1**) and (**BA2**) we obtain

$$\left\|\tilde{\Box}_g\partial\mathfrak{F} - \frac{1}{\lambda^2}g^{-1}(\mathrm{d}u_0,\mathrm{d}u_0)\partial\mathfrak{F}\right\|_{L^2} \lesssim \frac{1}{\lambda}.$$

Moreover, using (3.4.7) commuted with one derivative we obtain

$$\|\mathscr{L}_0\partial\mathfrak{F}\|_{L^2} \lesssim \|[\mathscr{L}_0,\partial]\mathfrak{F}\|_{L^2} + \|\partial\mathfrak{h}\|_{L^2} \lesssim 1$$

where we used (**BA1**) and (**BA3**). We proved that

$$\left\|\tilde{\Box}_g\left(\sin\left(\frac{u_0}{\lambda}\right)\partial\mathfrak{F}\right)\right\|_{L^2} \lesssim \frac{1}{\lambda}. \tag{3.7.22}$$

Together with (3.7.21), (3.7.22) proves that $\left\|\tilde{\Box}_g\Upsilon^\rho\right\|_{L^2} \lesssim \frac{1}{\lambda}$, which concludes the proof. □

3.7.3 Propagation of the second polarization

In this section, we prove that $V^{(2,i)} = 0$ on the whole spacetime. We start with the following lemma.

Lemma 3.7.7. *We have* $\mathcal{B}(V)_\alpha^{(-1)} = 0$.

Proof. The $\frac{1}{\lambda}$ terms in $\mathrm{div}_g G(V)_\alpha$ comes from the differenciation of the oscillating parts of $G^{(0)}_{\nu\alpha}$, i.e

$$\begin{aligned}\mathcal{B}(V)_\alpha^{(-1)} = &-\frac{1}{2}\cos\left(\frac{u_0}{\lambda}\right)\partial^\nu u_0\left(\partial_{(\alpha}u_0 V^{(2,1)}_{\nu)} + V^{(2,1)}_{L_0}(g_0)_{\alpha\nu}\right)\\ &+ 4\sin\left(\frac{2u_0}{\lambda}\right)\partial^\nu u_0\left(\partial_{(\alpha}u_0 V^{(2,2)}_{\nu)} + V^{(2,2)}_{L_0}(g_0)_{\alpha\nu}\right)\end{aligned}$$

where we recall (3.7.15). Now If $i = 1,2$, we have

$$\partial^\nu u_0\left(\partial_{(\alpha}u_0 V^{(2,i)}_{\nu)} + V^{(2,i)}_{L_0}(g_0)_{\alpha\nu}\right) = -\partial_\nu u_0 V^{(2,i)}_{L_0} + V^{(2,i)}_{L_0}\partial_\nu u_0 = 0$$

which concludes the proof. □

In order to propagate the polarization condition $V^{(2,i)} = 0$, we compute $\mathcal{B}_\alpha^{(0)}$.

Lemma 3.7.8. *We have*

$$\mathcal{B}(V)^{(0)}_\alpha = \frac{1}{2}\sin\left(\frac{u_0}{\lambda}\right)\left(-\mathscr{L}_0 V^{(2,1)}_\alpha + g_0^{\mu\nu}V^{(2,1)}_\nu \mathbf{D}_{[\mu}(L_0)_{\alpha]}\right)$$
$$+ 2\cos\left(\frac{2u_0}{\lambda}\right)\left(-\mathscr{L}_0 V^{(2,2)}_\alpha + g_0^{\mu\nu}V^{(2,2)}_\nu \mathbf{D}_{[\mu}(L_0)_{\alpha]}\right).$$

Proof. From the definition (3.7.18) we obtain

$$\mathrm{div}_g G(V)_\alpha = g^{\mu\nu}\partial_\mu G(V)_{\nu\alpha} - g^{\mu\nu}\Gamma^\rho_{\mu\nu}G(V)_{\rho\alpha} - g^{\mu\nu}\Gamma^\rho_{\mu\alpha}G(V)_{\rho\nu} \tag{3.7.23}$$

where $\Gamma^\rho_{\mu\nu}$ denotes the Christoffel symbols associated to g, which admits the following expansion

$$\Gamma^\rho_{\mu\nu} = \Gamma(g_0)^\rho_{\mu\nu} + (\tilde{\Gamma}^{(0)})^\rho_{\mu\nu} + O(\lambda)$$

where

$$(\tilde{\Gamma}^{(0)})^\rho_{\mu\nu} = -\frac{1}{2}\sin\left(\frac{u_0}{\lambda}\right) g_0^{\rho\sigma}\left(\partial_\mu u_0 F^{(1)}_{\nu\sigma} + \partial_\nu u_0 F^{(1)}_{\mu\sigma} - \partial_\sigma u_0 F^{(1)}_{\mu\nu}\right). \tag{3.7.24}$$

Note that $F^{(1)}_{L_0\alpha} = 0$ and $\mathrm{tr}_{g_0}F^{(1)} = 0$ imply $g_0^{\mu\nu}(\tilde{\Gamma}^{(0)})^\rho_{\mu\nu} = 0$. This gives

$$\mathcal{B}(V)^{(0)}_\alpha = g_0^{\mu\nu}\partial_\mu u_0 \partial_\theta G^{(1)}_{\nu\alpha} + g_0^{\mu\nu}\mathbf{D}_\mu G^{(0)}_{\nu\alpha} - g_0^{\mu\nu}(\tilde{\Gamma}^{(0)})^\rho_{\mu\alpha}G^{(0)}_{\rho\nu} \tag{3.7.25}$$

where in $\mathbf{D}_\mu G^{(0)}$ we don't differenciate the oscillating functions. We start by looking at the term depending on $G^{(1)}$, whose expression is given by (3.7.16). Thanks to $F^{(1)}_{L_0\alpha} = 0$ the terms proportional to $O\left(V^{(2)}\right)$ in (3.7.16) don't contribute to $\mathcal{B}(V)^{(0)}_\alpha$. We are left with

$$g_0^{\mu\nu}\partial_\mu u_0 \partial_\theta G^{(1)}_{\nu\alpha} = \frac{1}{2}\sin\left(\frac{u_0}{\lambda}\right)\left(L_0^\nu \mathbf{D}_{(\alpha}V^{(2,1)}_{\nu)} - \mathrm{div}_{g_0}V^{(2,1)}(L_0)_\alpha\right) \tag{3.7.26}$$
$$+ 2\cos\left(\frac{2u_0}{\lambda}\right)\left(L_0^\nu \mathbf{D}_{(\alpha}V^{(2,2)}_{\nu)} - \mathrm{div}_{g_0}V^{(2,2)}(L_0)_\alpha\right).$$

Let us now look at the third term in (3.7.25). Let $i = 1, 2$, we expand using (3.7.24):

$$(g_0)^{\mu\nu}(\tilde{\Gamma}^{(0)})^\rho_{\mu\alpha}\left(\partial_{(\rho}u_0 V^{(2,i)}_{\nu)} + V^{(2,i)}_{L_0}(g_0)_{\rho\nu}\right) \tag{3.7.27}$$
$$= -\frac{1}{2}\sin\left(\frac{u_0}{\lambda}\right) g_0^{\mu\nu}g_0^{\rho\sigma}\left(\partial_{(\mu}u_0 F^{(1)}_{\sigma\alpha)} - \partial_\sigma u_0 F^{(1)}_{\mu\alpha}\right)\partial_{(\rho}u_0 V^{(2,i)}_{\nu)}$$
$$- \frac{1}{2}\sin\left(\frac{u_0}{\lambda}\right) g_0^{\mu\nu}g_0^{\rho\sigma}\left(\partial_{(\mu}u_0 F^{(1)}_{\sigma\alpha)} - \partial_\sigma u_0 F^{(1)}_{\mu\alpha}\right)V^{(2,i)}_{L_0}(g_0)_{\rho\nu}$$
$$= -\frac{1}{2}\sin\left(\frac{u_0}{\lambda}\right) g_0^{\mu\nu}g_0^{\rho\sigma}\left(\partial_{(\mu}u_0 F^{(1)}_{\sigma\alpha)} - \partial_\sigma u_0 F^{(1)}_{\mu\alpha}\right)\partial_\nu u_0 V^{(2,i)}_\rho$$
$$= 0.$$

Therefore the third term in (3.7.25) vanishes. Finally, let us look at the second term in this expression. From $\mathbf{D}g_0 = 0$ we obtain

$$g_0^{\mu\nu}\mathbf{D}_\mu G^{(0)}_{\nu\alpha} = -\frac{1}{2}\sin\left(\frac{u_0}{\lambda}\right)\left(-g_0^{\mu\nu}V^{(2,1)}_\nu \mathbf{D}_\mu(L_0)_\alpha - V^{(2,1)}_\alpha \mathrm{div}_{g_0}L_0\right. \tag{3.7.28}$$
$$\left. -(L_0)_\alpha \mathrm{div}_{g_0}V^{(2,1)} - \mathbf{D}_{L_0}V^{(2,1)}_\alpha + \partial_\alpha V^{(2,1)}_{L_0}\right)$$
$$- 2\cos\left(\frac{2u_0}{\lambda}\right)\left(-g_0^{\mu\nu}V^{(2,2)}_\nu \mathbf{D}_\mu(L_0)_\alpha - V^{(2,2)}_\alpha \mathrm{div}_{g_0}L_0\right.$$
$$\left. -(L_0)_\alpha \mathrm{div}_{g_0}V^{(2,2)} - \mathbf{D}_{L_0}V^{(2,1)}_\alpha + \partial_\alpha V^{(2,2)}_{L_0}\right).$$

Adding this to (3.7.26) concludes the proof. □

Before proving the propagation of the second polarization, we first prove that $V^{(2,i)}_{\underline{L}_0} = 0$ on Σ_0. Recall that the tangential components of $V^{(2,i)}$ already vanish on Σ_0 thanks to our choice of initial data for $F^{(2,i)}$, see (3.4.46). Since $V^{(2,i)}_{\underline{L}_0}$ only involves the projection of the metric on Σ_0 (see (3.2.11)), $V^{(2,i)}_{\underline{L}_0} = 0$ can't be ensured by well-chosen initial data, it has to be directly satisfied by the solution of the constraint equations. This means that we could have performed this computation in Chapter 2. However, it seems more consistent with the splitting between Chapters 2 and 3 of this thesis to study $V^{(2,i)}_{\underline{L}_0}$ here. Moreover, we will benefit from the reduced form of the Einstein tensor obtained in Lemma 3.7.5, and the fact that the constraint equations are solved on the initial hypersurface.

Lemma 3.7.9. *We have*

$$V^{(2,i)} \upharpoonright \Sigma_0 = 0.$$

Proof. Thanks to (3.4.28), it remains to show that $V^{(2,i)}_{\underline{L}_0} = 0$ on Σ_0. In fact, it is enough to show that

$$V^{(2,i)}_0 \upharpoonright \Sigma_0 = 0. \tag{3.7.29}$$

Indeed, if (3.7.29) holds then $V^{(2,i)}_{\underline{L}_0} = -V^{(2,i)}_{L_0}$ (where we used that $L_0 = |\nabla u_0|_{\bar{g}_0}(\partial_t + N_0)$ and $\underline{L}_0 = |\nabla u_0|_{\bar{g}_0}(\partial_t - N_0)$ on Σ_0) and (3.4.28) then gives the result. Therefore, let us prove (3.7.29).

Lemma 3.7.5 and (3.4.28) imply that on Σ_0

$$G(V)_{\alpha\beta} = -\frac{1}{2}\sin\left(\frac{u_0}{\lambda}\right)\partial_{(\alpha}u_0 V^{(2,1)}_{\beta)} - 2\cos\left(\frac{2u_0}{\lambda}\right)\partial_{(\alpha}u_0 V^{(2,2)}_{\beta)} + O(\lambda)\,.$$

Moreover, Υ^ρ contains oscillating terms (see (3.3.16)) and also terms depending on λ like $\mathfrak{h}$ or $\mathfrak{F}$. More precisely, if we only write the top derivatives of the non-background terms we have

$$|G(\Upsilon)| \lesssim \lambda|\partial\mathfrak{F}| + \lambda^2\left(|\partial^2\mathfrak{F}| + |\partial^2\mathfrak{h}|\right).$$

Estimates (**BA1**) and (**BA3**) then imply that $G(\Upsilon)$ is $O(\lambda)$ in $L^2(K)$. Conclusion, we have

$$G_{\alpha\beta} = -\frac{1}{2}\sin\left(\frac{u_0}{\lambda}\right)\partial_{(\alpha}u_0 V^{(2,1)}_{\beta)} - 2\cos\left(\frac{2u_0}{\lambda}\right)\partial_{(\alpha}u_0 V^{(2,2)}_{\beta)} + O(\lambda) \tag{3.7.30}$$

where the $O(\lambda)$ has to be understood in $L^2(K)$. Now, Corollary 3.4.1 ensures that the constraint equations are satisfied on Σ_0, which in particular gives $G_{TT} = 0$ on Σ_0, where T is the unit normal to Σ_0 for g. Thanks to Lemma 3.4.2, we have $T = \partial_t + O(\lambda^2)$ so the previous identity implies that $G_{00} = O(\lambda^2)$ on Σ_0. Thanks to (3.7.30) and (3.1.11) this rewrites as

$$-\sin\left(\frac{u_0}{\lambda}\right)|\nabla u_0|_{\bar{g}_0}V^{(2,1)}_0 - 4\cos\left(\frac{2u_0}{\lambda}\right)|\nabla u_0|_{\bar{g}_0}V^{(2,2)}_0 = O(\lambda)\,. \tag{3.7.31}$$

We multiply this identity by $\sin\left(\frac{u_0}{\lambda}\right)$ and take the weak limit in $L^2(K)$ when λ tends to 0. Remark 3.7.1 shows that the weak limit of $\sin^2\left(\frac{u_0}{\lambda}\right)$ and $\sin\left(\frac{u_0}{\lambda}\right)\cos\left(\frac{2u_0}{\lambda}\right)$ are respectively $\frac{1}{2}$ and 0. Moreover $V^{(2,1)}_0$ does not depend on λ so we obtain $V^{(2,1)}_0 = 0$. Then (3.7.31) becomes

$$\cos\left(\frac{2u_0}{\lambda}\right)|\nabla u_0|_{\bar{g}_0}V^{(2,2)}_0 = O(\lambda)\,. \tag{3.7.32}$$

Multiplying this identity by $\cos\left(\frac{2u_0}{\lambda}\right)$ and taking the weak limit in $L^2(K)$ when λ tends to 0 gives similarly that $V^{(2,2)}_0 = 0$. This proves (3.7.29) and concludes the proof of the lemma. □

We can now prove the propagation of the second polarization.

Proposition 3.7.1. *The following holds on the whole spacetime:*

$$V^{(2,i)} = 0 \tag{3.7.33}$$

for $i = 1, 2$.

Proof. We look at the quantity $\sin\left(\frac{u_0}{\lambda}\right)\mathrm{div}_g G_\alpha$, which thanks to Lemmas 3.7.7 and 3.7.8 rewrites

$$\begin{aligned}\sin\left(\frac{u_0}{\lambda}\right)\mathrm{div}_g G_\alpha &= \frac{1}{2}\sin^2\left(\frac{u_0}{\lambda}\right)\left(-\mathscr{L}_0 V^{(2,1)}_\alpha + g_0^{\mu\nu}V^{(2,1)}_\nu \mathbf{D}_{[\mu}(L_0)_{\alpha]}\right)\\ &\quad + 2\sin\left(\frac{u_0}{\lambda}\right)\cos\left(\frac{2u_0}{\lambda}\right)\left(-\mathscr{L}_0 V^{(2,2)}_\alpha + g_0^{\mu\nu}V^{(2,2)}_\nu \mathbf{D}_{[\mu}(L_0)_{\alpha]}\right)\\ &\quad + O(\lambda)\end{aligned} \tag{3.7.34}$$

where we use the Bianchi identities (3.7.17), Lemma 3.7.6 and where the $O(\lambda)$ has to be understood in $L^2(K)$. The quantity

$$-\mathscr{L}_0 V^{(2,1)}_\alpha + g_0^{\mu\nu}V^{(2,1)}_\nu \mathbf{D}_{[\mu}(L_0)_{\alpha]}$$

only involves the background perturbations so it doesn't depend on λ. Therefore the weak limit when λ tends to 0 of the first two lines in (3.7.34) is equal to

$$\frac{1}{4}\left(-\mathscr{L}_0 V^{(2,1)}_\alpha + g_0^{\mu\nu}V^{(2,1)}_\nu \mathbf{D}_{[\mu}(L_0)_{\alpha]}\right). \tag{3.7.35}$$

Thanks to (3.7.17), the weak limit of (3.7.34) is zero and we obtain the following equation for $V^{(2,1)}$:

$$\mathscr{L}_0 V^{(2,1)}_\alpha = g_0^{\mu\nu}V^{(2,1)}_\nu \mathbf{D}_{[\mu}(L_0)_{\alpha]}. \tag{3.7.36}$$

If we now multiply $\mathrm{div}_g G_\alpha$ by $\cos\left(\frac{2u_0}{\lambda}\right)$, Lemma 3.7.8 and (3.7.36) imply

$$\cos\left(\frac{2u_0}{\lambda}\right)\mathcal{B}_\alpha = 2\cos^2\left(\frac{2u_0}{\lambda}\right)\left(-\mathscr{L}_0 V^{(2,2)}_\alpha + g_0^{\mu\nu}V^{(2,2)}_\nu \mathbf{D}_{[\mu}(L_0)_{\alpha]}\right) + O(\lambda).$$

We again take the weak limit in L^2 when λ tends to 0 and thanks to (3.7.17) we obtain

$$\mathscr{L}_0 V^{(2,2)}_\alpha = g_0^{\mu\nu}V^{(2,2)}_\nu \mathbf{D}_{[\mu}(L_0)_{\alpha]}. \tag{3.7.37}$$

Equations (3.7.36) and (3.7.37) are transport equations for the quantities $V^{(2,i)}$, $i = 1, 2$. Thanks to Lemma 3.7.9 we know that

$$V^{(2,i)} \restriction \Sigma_0 = 0 \tag{3.7.38}$$

for $i = 1, 2$. Together with the energy estimate for the transport operator L_0 (see Lemma 3.5.1) and Gronwall's inequality, (3.7.37) and (3.7.38) lead to (3.7.33).

□

3.7.4 Propagation of the generalised wave gauge and conclusion

In the previous sections, we proved that $V^{(2,i)} = 0$ for $i = 1, 2$. In order to prove that g is a solution of the Einstein vacuum equations, it remains to prove that $\Upsilon^\rho = 0$. Indeed, the Einstein tensor of g now reduces to $G(\Upsilon)$, i.e

$$G_{\alpha\beta} = \frac{\lambda^2}{2}\left(g_{\rho(\alpha}\partial_{\beta)}\Upsilon^\rho - g_{\alpha\beta}\partial_\rho\Upsilon^\rho + \Upsilon^\rho\partial_\rho g_{\alpha\beta} - \frac{1}{2}g^{\mu\nu}\Upsilon^\rho\partial_\rho g_{\mu\nu}g_{\alpha\beta}\right) \tag{3.7.39}$$

where in $\partial_{\beta)}\Upsilon^\rho$ we also differenciate the oscillatory parts of Υ^ρ. At this point, it is standard to deduce from the Bianchi identities (3.7.17) the fact that Υ^ρ solves a system of linear wave system on the spacetime.

Proposition 3.7.2. *The following holds on the whole spacetime:*

$$\Upsilon^\rho = 0.$$

Proof. The main ingredient is already contained in (3.7.19). Since $V^{(2,i)} = 0$, we have $G_{\alpha\beta} = G(\Upsilon)_{\alpha\beta}$ and (3.7.19) together with (3.7.17) then implies

$$\tilde{\Box}_g\Upsilon^\rho + A^{\mu\rho}_\nu\partial_\mu\Upsilon^\nu + B^\rho_\alpha\Upsilon^\alpha = 0. \tag{3.7.40}$$

As announced, Υ solves (3.7.40), i.e a system of linear wave equations on the spacetime. If Υ^ρ and $T\Upsilon^\rho$ vanish on Σ_0, then Υ^ρ vanishes on the whole spacetime (recall that T is the unit normal to Σ_0 for g, see Lemma 3.4.2). From Corollary 3.4.1 and our choice of initial data for g, we deduce what we need.

- Our choice of initial data for $\partial_t\mathfrak{h}_{0\mu}$ implies that $\Upsilon^\rho \restriction \Sigma_0 = 0$, see the end of Section 3.4.6.2 or Corollary 3.4.1. Thanks to (3.7.39) this also implies that on Σ_0 we have

$$G_{\alpha\beta} = \frac{\lambda^2}{2}\left(g_{\rho(\alpha}\partial_{\beta)}\Upsilon^\rho - g_{\alpha\beta}\partial_t\Upsilon^0\right) \tag{3.7.41}$$

- Corollary 3.4.1 states that the constraint equations are solved on Σ_0, which reads $G_{TT} = 0$ and $G_{Ti} = 0$. Since $T = (1 + Z^t)\partial_t + \bar{Z}^i\partial_i$ and $g(T, \partial_i) = 0$ the previous components of the Einstein tensor rewrite

$$G_{TT} = 2T^\alpha g_{\rho\alpha}T\Upsilon^\rho + \partial_t\Upsilon^0 = -\frac{T\Upsilon^0}{1 + Z^t}$$

  and

$$G_{Ti} = g_{0i}T\Upsilon^0 + \bar{g}_{ij}T\Upsilon^j$$

  where we used (3.7.41) and $\Upsilon^\rho \restriction \Sigma_0 = 0$ to cancel any spatial derivative of Υ^ρ. Since $Z^t = O\left(\lambda^4\right)$ (see (3.4.34)) the constraint equations indeed imply that $T\Upsilon^\rho \restriction \Sigma_0 = 0$.

□

Propositions 3.7.1 and 3.7.2 concludes the proof of Theorem 3.1.2, by showing that the metric g solution of the background and reduced systems is actually a solution of the Einstein vacuum equations.

Appendix

3.A The commutator $[\Box_{g_0}, L_0]$ and proof of Lemma 3.2.1

This first appendix is devoted to the proof of Lemma 3.2.1, which estimates the commutator $[\Box_{g_0}, L_0]$, where L_0 is defined by (3.1.5) and g_0 is the background metric.

A general computation using the expression of the wave operator in coordinates shows that if X is any vector field and f a scalar function we have

$$\begin{aligned}[\Box_{g_0}, X]f &= 2g_0^{\alpha\beta}g_0^{\mu\nu}(\mathbf{D}_\alpha X)_\mu\partial_\beta\partial_\nu f + \partial_\alpha f\left(\Box_{g_0}X^\alpha + X\left(g_0^{\rho\sigma}\Gamma(g_0)^\alpha_{\rho\sigma}\right)\right)\\ &= 2g_0^{\alpha\beta}g_0^{\mu\nu}(\mathbf{D}_\alpha X)_\mu\partial_\beta\partial_\nu f + \partial_\alpha f\Box_{g_0}X^\alpha \qquad (3.A.1)\end{aligned}$$

where we also used the wave coordinate condition for the background (3.1.6). In the context of Lemma 3.2.1, we consider $X = L_0$ and f compactly supported and we are only interested in the principal terms in $[\Box_{g_0}, L_0]f$. Indeed the non-principal part simply satisfies

$$|\partial_\alpha f\Box_{g_0}L_0^\alpha| \lesssim |\partial f|. \qquad (3.A.2)$$

with a constant only depending on the background spacetime. It remains to estimate the second derivatives of f in (3.A.1), i.e

$$g_0^{\alpha\beta}g_0^{\mu\nu}(\mathbf{D}_\alpha L_0)_\mu\partial_\beta\partial_\nu f,$$

by using the expression of the background metric in the background null frame defined in Section 3.1.2:

$$g_0^{\alpha\beta} = -\frac{1}{2}L_0^{(\alpha}\underline{L}_0^{\beta)} + \delta^{AB}e_A^\alpha e_B^\beta. \qquad (3.A.3)$$

Using (3.A.3) twice and the fact that L_0 is geodesic, that is $\mathbf{D}_{L_0}L_0 = 0$, we obtain

$$\begin{aligned}g_0^{\alpha\beta}g_0^{\mu\nu}(\mathbf{D}_\alpha L_0)_\mu\partial_\beta\partial_\nu f &= -\frac{1}{2}g_0^{\mu\nu}(\mathbf{D}_{L_0}L_0)_\mu\partial_\nu\underline{L}_0 f - \frac{1}{2}g_0^{\mu\nu}(\mathbf{D}_{\underline{L}_0}L_0)_\mu\partial_\nu L_0 f\\ &\quad + \delta^{AB}g_0^{\mu\nu}(\mathbf{D}_{e_A}L_0)_\mu\partial_\nu e_B f\\ &= -\frac{1}{2}g_0^{\mu\nu}(\mathbf{D}_{\underline{L}_0}L_0)_\mu\partial_\nu L_0 f - \frac{1}{2}\delta^{AB}L_0^\mu(\mathbf{D}_{e_A}L_0)_\mu\underline{L}_0 e_B f\\ &\quad - \frac{1}{2}\delta^{AB}\underline{L}_0^\mu(\mathbf{D}_{e_A}L_0)_\mu L_0 e_B f + \delta^{AB}\delta^{CD}e_C^\mu(\mathbf{D}_{e_A}L_0)_\mu e_D e_B f\\ &= -\frac{1}{2}g_0^{\mu\nu}(\mathbf{D}_{\underline{L}_0}L_0)_\mu\partial_\nu L_0 f - \frac{1}{2}\delta^{AB}\underline{L}_0^\mu(\mathbf{D}_{e_A}L_0)_\mu L_0 e_B f\\ &\quad + \delta^{AB}\delta^{CD}e_C^\mu(\mathbf{D}_{e_A}L_0)_\mu e_D e_B f\end{aligned}$$

where we also used $L_0^\mu(\mathbf{D}_{e_A} L_0)_\mu = 0$, which follows from the fact $g_0(L_0, L_0) = 0$ is constant. Using now the fact that

$$|L_0 e_B f| \lesssim |\partial L_0 f| + |\partial f|$$

and recalling (3.A.1) and (3.A.2) we obtain

$$\|[\Box_{g_0}, L_0]f\|_{L^2} \lesssim \|e_A e_B f\|_{L^2} + \|\partial L_0 f\|_{L^2} + \|\partial f\|_{L^2} . \tag{3.A.4}$$

where we also used the compact support of f to switch from pointwise estimates to L^2 norms. Therefore, in order to prove Lemma 3.2.1 it remains to estimate $\|e_A e_B f\|_{L^2}$.

This follows from elliptic estimates on $P_{t,u}$. If K denotes the Gauss curvature of $P_{t,u}$ and if $\mathrm{d}\mu_{t,u}$ denotes the volume form on $P_{t,u}$ induced by $\mathring{g}_0$ (the induced metric on $P_{t,u}$), then the scalar Bochner identity (Proposition 3.5 in [Sze18]) reads

$$\int_{P_{t,u}} |\mathrm{Hess} f|^2_{\mathring{g}_0} \,\mathrm{d}\mu_{t,u} = \int_{P_{t,u}} (\not{\Delta} f)^2 \mathrm{d}\mu_{t,u} - \int_{P_{t,u}} K |\not{\nabla} f|^2_{\mathring{g}_0} \mathrm{d}\mu_{t,u}. \tag{3.A.5}$$

where

$$\begin{aligned} \not{\Delta} f &= \delta^{AB} e_A e_B f + \delta^{AB}\delta^{CD} g(\not{\nabla}_{e_A} e_C, e_B) e_D f, \\ \not{\nabla} f &= \delta^{AB}(e_A f) e_B. \end{aligned}$$

Thanks to the regularity assumptions stated in Section 3.1.2 and the fact that the 2-surfaces $P_{t,u}$ foliates Σ_t, we can integrate (3.A.5) in the N_0 direction and obtain

$$\|e_A e_B f\|_{L^2} \lesssim \|\not{\Delta} f\|_{L^2} + \|\partial f\|_{L^2}$$

where we recall that $\|\cdot\|_{L^2}$ denotes $\|\cdot\|_{L^2(\Sigma_t)}$ and where the implicit constant depends only on background quantities. Together with (3.A.4) this gives

$$\|[\Box_{g_0}, L_0]f\|_{L^2} \lesssim \|\not{\Delta} f\|_{L^2} + \|\partial L_0 f\|_{L^2} + \|\partial f\|_{L^2} . \tag{3.A.6}$$

We now use Lemma 2.5 of [Sze12] which gives the expression of the wave operator in the null frame and implies

$$\|\not{\Delta} f\|_{L^2} \lesssim \|\Box_{g_0} f\|_{L^2} + \|\partial L_0 f\|_{L^2} + \|\partial f\|_{L^2} .$$

Together with (3.A.6) this concludes the proof of the first part of Lemma 3.2.1.

The second part is proved in a similar way. For $r \geq 1$, we apply (3.2.25) to $\nabla^r f$ and obtain

$$\begin{aligned} \|[L_0, \Box_{g_0}]\nabla^r f\|_{L^2} &\lesssim \|\partial L_0 \nabla^r f\|_{L^2} + \|\Box_{g_0} \nabla^r f\|_{L^2} + \|\partial f\|_{H^r} \\ &\lesssim \|\nabla^r \partial L_0 f\|_{L^2} + \|\nabla^r \Box_{g_0} f\|_{L^2} + \|\partial f\|_{H^r} \\ &\quad + \|[\partial L_0, \nabla^r] f\|_{L^2} + \|[\Box_{g_0}, \nabla^r] f\|_{L^2} \end{aligned}$$

Moreover we have

$$\|[\partial L_0, \nabla^r] f\|_{L^2} + \|[\Box_{g_0}, \nabla^r] f\|_{L^2} \lesssim \left\|\partial^2 f\right\|_{H^{r-1}}$$

where we recall that $r \geq 1$. It remains to notice that

$$\|\nabla^r [L_0, \Box_{g_0}] f\|_{L^2} \lesssim \|[L_0, \Box_{g_0}]\nabla^r f\|_{L^2} + \|\partial f\|_{H^r} + \left\|\partial^2 f\right\|_{H^{r-1}} .$$

This concludes the proof of Lemma 3.2.1.

3.B The spectral projections

In this section, we prove Lemma 3.2.2. The proof is based on the dyadic decomposition at the heart of Littlewood-Paley theory, which we present shortly.

3.B.1 Littlewood-Paley theory

This presentation of the Littlewood-Paley theory is based on [BCD11]. We start by considering two smooth radial functions χ and φ from $\mathbb{R}^3$ to the interval $[0,1]$ supported in $\left\{|\xi| \leq \frac{4}{3}\right\}$ and $\left\{\frac{3}{4} \leq |\xi| \leq \frac{8}{3}\right\}$ respectively and such that

$$\chi + \sum_{j\geq 0} \varphi\left(2^{-j}\cdot\right) = 1, \tag{3.B.1}$$

$$|j-i| \geq 2 \Longrightarrow \operatorname{supp}\left(\varphi\left(2^{-j}\cdot\right)\right) \cap \operatorname{supp}\left(\varphi\left(2^{-i}\cdot\right)\right) = \emptyset,$$

$$j \geq 1 \Longrightarrow \operatorname{supp}(\chi) \cap \operatorname{supp}\left(\varphi\left(2^{-j}\cdot\right)\right) = \emptyset,$$

$$\frac{1}{2} \leq \chi^2 + \sum_{j\geq 0} \varphi^2\left(2^{-j}\cdot\right) \leq 1. \tag{3.B.2}$$

The existence of χ and φ is the content of Proposition 2.10 in [BCD11]. They define a dyadic partition of unity, with which we can define the dyadic blocks Δ_j. If u is a tempered distribution on $\mathbb{R}^3$ we set $\Delta_i u = 0$ if $i \leq -2$ and

$$\begin{aligned} \Delta_{-1} u &= \mathcal{F}^{-1}\left(\chi\mathcal{F}(u)\right) \\ \Delta_j u &= \mathcal{F}^{-1}\left(\varphi\left(2^{-j}\cdot\right)\mathcal{F}(u)\right) \end{aligned}$$

for $j \geq 0$ and where $\mathcal{F}$ denotes the Fourier transform on $\mathbb{R}^3$. We also define

$$S_j u = \sum_{j' \leq j-1} \Delta_{j'} u.$$

for $j \geq 0$. For u a tempered distribution, we use the notation $\operatorname{spect}(u) = \operatorname{supp}(\mathcal{F}(u))$. Therefore the support properties of χ and φ imply

$$\operatorname{spect}(\Delta_{-1} u) \subset \left\{|\xi| \leq \frac{4}{3}\right\}, \tag{3.B.3}$$

$$\operatorname{spect}(\Delta_j u) \subset 2^j \left\{\frac{3}{4} \leq |\xi| \leq \frac{8}{3}\right\} \tag{3.B.4}$$

for $j \geq 0$. The following proposition contains all the basic estimates we need on the dyadic blocks.

Proposition 3.B.1. *Let u be a tempered distribution.*

1. *The so-called Bernstein estimates hold*

$$\|X_j \nabla^\alpha u\|_{L^p} \lesssim 2^{\alpha j} \|X_j u\|_{L^p} \tag{3.B.5}$$

for $X_j \in \{S_j, \Delta_j\}$, $\alpha = 0, 1$ and for all $1 \leq p \leq +\infty$.

2. *Moreover, the following estimates hold*

$$\|X_j u\|_{L^\infty} \lesssim 2^{\frac{3}{2}j} \|X_j u\|_{L^2}. \tag{3.B.6}$$

for $X_j \in \{S_j, \Delta_j\}$.

Proof. For $j \geq -1$, we have $\operatorname{spect}(X_j \nabla u) \subset 2^j B$ for B a ball centered at 0 in $\mathbb{R}^3$. Let ϕ be a compactly supported function on $\mathbb{R}^3$ such that $\phi \restriction B = 1$. This implies

$$\mathcal{F}(X_j u) = \phi\left(2^{-j}\cdot\right)\mathcal{F}(X_j u).$$

Therefore, for $\alpha = 0, 1$ we have

$$X_j \nabla^\alpha u = 2^{(3+\alpha)j}\left(\nabla^\alpha \mathcal{F}^{-1}(\phi)\right)\left(2^j\cdot\right) * X_j u \tag{3.B.7}$$

where the symbol $*$ denotes the convolution between two functions. The Young inequality implies

$$\begin{aligned}\|X_j \nabla^\alpha u\|_{L^p} &\lesssim 2^{(3+\alpha)j}\left\|\left(\nabla^\alpha \mathcal{F}^{-1}(\phi)\right)\left(2^j\cdot\right)\right\|_{L^1}\|X_j u\|_{L^p}\\ &\lesssim 2^{\alpha j}\left\|\nabla^\alpha \mathcal{F}^{-1}(\phi)\right\|_{L^1}\|X_j u\|_{L^p}\end{aligned}$$

where we change variable in the last step. This proves (3.B.5). Now, if $\alpha = 0$ in (3.B.7), another case of Young inequality implies

$$\begin{aligned}\|X_j u\|_{L^\infty} &\lesssim 2^{3j}\left\|\mathcal{F}^{-1}(\phi)\left(2^j\cdot\right)\right\|_{L^2}\|X_j u\|_{L^2}\\ &\lesssim 2^{\frac{3}{2}j}\left\|\mathcal{F}^{-1}(\phi)\right\|_{L^2}\|X_j u\|_{L^2}\end{aligned}$$

which concludes the proof of the proposition. □

The property (3.B.1) allows us to decompose each function u into an infinite sum of smooth and localized in Fourier space functions:

$$u = \sum_{j\geq -1}\Delta_j u. \tag{3.B.8}$$

The series (3.B.8) is *a priori* purely formal, and though each dyadic blocks $\Delta_j u$ is smooth the function u might have to be defined merely as a distribution. However, one can link the regularity of u and the summability properties of the series. The next proposition gives a characterization of the Sobolev spaces in terms of the convergence of the series (3.B.8).

Proposition 3.B.2. *If* $s \in \mathbb{R}$ *and* u *is a tempered distribution, then*

$$u \in H^s \iff \left(2^{js}\|\Delta_j u\|_{L^2}\right)_{j\in\mathbb{N}} \in \ell^2(\mathbb{N}).$$

Moreover, there exists $C > 0$ *such that for all tempered distribution* u *we have*

$$\frac{1}{C}\sum_{j\geq -1}2^{2js}\|\Delta_j u\|^2_{L^2} \leq \|u\|^2_{H^s} \leq C\sum_{j\geq -1}2^{2js}\|\Delta_j u\|^2_{L^2}. \tag{3.B.9}$$

Proof. By definition of the space H^s, we have $\|u\|_{H^s} = \|\langle\cdot\rangle^s\mathcal{F}(u)\|_{L^2}$. Using the first inequality in (3.B.2) we obtain

$$\begin{aligned}\|u\|^2_{H^s} &\leq 2\left(\int_{\mathbb{R}^3}\langle\xi\rangle^{2s}\chi^2(\xi)|\mathcal{F}(u)(\xi)|^2\mathrm{d}\xi + \sum_{j\geq 0}\int_{\mathbb{R}^3}\langle\xi\rangle^{2s}\varphi^2\left(2^{-j}\xi\right)|\mathcal{F}(u)(\xi)|^2\mathrm{d}\xi\right)\\ &\lesssim 2^{-2s}\|\Delta_{-1}u\|^2_{L^2} + \sum_{j\geq 0}2^{2js}\|\Delta_j u\|^2_{L^2}\end{aligned}$$

where we used the support properties of χ and φ. This proves half of (3.B.9), the other half is proved similarly using the other inequality in (3.B.2). □

The decomposition (3.B.8) is also very useful to study products of functions. Indeed, formally we have

$$uv = \sum_{j,j' \geq -1} \Delta_j u \Delta_{j'} v$$

for u and v two tempered distributions. As it is well known the product uv could not even be a tempered distribution. In order to understand the product's properties, Bony introduced in [Bon81] what is now called the Bony decomposition:

Definition 3.B.1. *For u and v two tempered distributions, we define the paraproduct $T_u v$ and the remainder $R(u,v)$ by*

$$T_u v = \sum_j S_{j-1} u \Delta_j v,$$
$$R(u,v) = \sum_{|j-k| \leq 1} \Delta_j u \Delta_k v.$$

The Bony decomposition of uv is then

$$uv = T_u v + T_v u + R(u,v).$$

In the sequel, we don't directly use Bony's decomposition but rather the spectral property of the general terms in the series defining $T_u v$ and $R(u,v)$, namely if $i \geq -1$ and $|j-k| \leq 1$:

$$\text{spect}\,(S_{i-1} u \Delta_i v) \subset 2^i \mathcal{C}', \tag{3.B.10}$$
$$\text{spect}\,(\Delta_j u \Delta_k v) \subset 2^j B', \tag{3.B.11}$$

with $\mathcal{C}' = \left\{ \frac{1}{12} \leq |\xi| \leq 3 \right\}$ and $B' = \left\{ 0 \leq |\xi| \leq \frac{32}{3} \right\}$.

3.B.2 Definition and first properties of $\Pi_{\leq}$ and $\Pi_{\geq}$

Recall Definition 3.4.1, where we define the operators $\Pi_{\leq}$ and $\Pi_{\geq}$. Their behaviour with respect to derivatives is similar to the one of dyadic blocks (see Proposition 3.B.1) thanks to the support property of their symbols.

Lemma 3.B.1. *Let f be a scalar function. We have*

$$\lambda \left\| \nabla \Pi_{\leq}(f) \right\|_{L^2} \lesssim \left\| f \right\|_{L^2}, \tag{3.B.12}$$
$$\left\| \Pi_{\geq}(f) \right\|_{L^2} \lesssim \lambda \left\| \nabla f \right\|_{L^2}. \tag{3.B.13}$$

Proof. To prove (3.B.12), we use Parseval's identity twice and the support property of χ_λ:

$$\left\| \nabla \Pi_{\leq}(f) \right\|_{L^2}^2 \leq \int_{|\xi| \leq \frac{2}{\lambda}} |\xi|^2 \left| \mathcal{F}(f)(\xi) \right|^2 \mathrm{d}\xi \lesssim \frac{1}{\lambda^2} \int_{\mathbb{R}^4} \left| \mathcal{F}(f)(\xi) \right|^2 \mathrm{d}\xi = \frac{1}{\lambda^2} \left\| f \right\|_{L^2}.$$

The proof of (3.B.12) is similar:

$$\left\| \Pi_{\geq}(f) \right\|_{L^2}^2 \leq \int_{\lambda|\xi| \geq 1} \left| \mathcal{F}(f)(\xi) \right|^2 \mathrm{d}\xi \lesssim \lambda^2 \int_{\mathbb{R}^4} |\xi|^2 \left| \mathcal{F}(f)(\xi) \right|^2 \mathrm{d}\xi = \lambda^2 \left\| \nabla f \right\|_{L^2}^2.$$

□

The following lemma is a special case of Lemma 2.97 in [BCD11]. For the sake of completeness, we redo the proof.

Lemma 3.B.2. *Let u and v be two tempered distribution. We have*

$$\left\| \left[u, \Pi_{\leq}\right] v \right\|_{L^2} \lesssim \lambda \left\| \nabla u \right\|_{L^\infty} \left\| v \right\|_{L^2}. \tag{3.B.14}$$

Proof. Using the usual notation of pseudo-differential calculus we have $\Pi_{\leq} = \chi_1(\lambda\nabla)$. For f a function we have $\chi_1(\lambda\nabla) f = \lambda^{-3}\left(\mathcal{F}^{-1}\chi_1\right)\left(\lambda^{-1}\cdot\right) * f$. This gives

$$\begin{aligned}
\left(\left[u, \chi_1(\lambda\nabla)\right] v\right)(x) &= \frac{1}{\lambda^3}\int_{\mathbb{R}^3} \left(\mathcal{F}^{-1}\chi_1\right)\left(\lambda^{-1}(x-y)\right)(u(y)-u(x))v(y)\mathrm{d}y \\
&= -\frac{1}{\lambda^3}\int_{[0,1]\times\mathbb{R}^3} \left(\mathcal{F}^{-1}\chi_1\right)\left(\lambda^{-1}z\right)\langle \nabla u(x-z\tau), z\rangle v(x-z)\mathrm{d}z\mathrm{d}\tau
\end{aligned}$$

where we used the Taylor expansion for formula for u. Defining $\Psi(z) = z\left(\mathcal{F}^{-1}\chi_1\right)(z)$ we obtain

$$\left|\left(\left[u, \chi_1(\lambda\nabla)\right] v\right)(x)\right| \leq \frac{\|\nabla u\|_{L^\infty}}{\lambda^2}\int_{[0,1]\times\mathbb{R}^3} \left|\Psi\left(\lambda^{-1}z\right)\right| |v(x-z)|\mathrm{d}z\mathrm{d}\tau.$$

We take the L^2 norm of this inequality to obtain after a last change of variable in Ψ:

$$\left\| \left[u, \chi_1(\lambda\nabla)\right] v \right\|_{L^2} \leq \lambda \|\nabla u\|_{L^\infty} \|v\|_{L^2} \|\Psi\|_{L^1}.$$

This concludes the proof, noting that $\|\Psi\|_{L^1} \lesssim \|\chi_1\|_{C^k}$ for k large enough. □

3.B.3 Proof of Lemma 3.2.2

Let $N \in \mathbb{N}$ the unique integer such that $\frac{1}{\lambda} \leq 2^N < \frac{2}{\lambda}$. In this proof, we will mainly use the fact that $\chi_\lambda(\xi) = 1$ if $|\xi| \leq 2^{N-1}$ and $\chi_\lambda(\xi) = 0$ if $|\xi| \geq 2^{N+1}$ (recall that χ_λ is defined in Definition 3.4.1). To benefit from this, we use the dyadic decompositions of u and v and regroup terms following the paraproduct decomposition (see Definition 3.B.1) to obtain:

$$\begin{aligned}
\left[u, \Pi_{\leq}\right] \nabla v &= \sum_j \left[S_{j-1}u, \Pi_{\leq}\right] \Delta_j \nabla v + \sum_j \left[\Delta_j u, \Pi_{\leq}\right] S_{j-1}\nabla v + \sum_{|i-j|\leq 1} \left[\Delta_i u, \Pi_{\leq}\right] \Delta_j \nabla v \\
&=: A_1 + A_2 + A_3.
\end{aligned}$$

We start with A_1. We have

$$\left[S_{j-1}u, \Pi_{\leq}\right] \Delta_j \nabla v = S_{j-1}u \Pi_{\leq}\left(\Delta_j \nabla v\right) - \Pi_{\leq}\left(S_{j-1}u \Delta_j \nabla v\right). \tag{3.B.15}$$

If j is such that $2^j \times \frac{1}{12} \geq 2^{N+1}$ then (3.B.4) implies $\Pi_{\leq}\Delta_j = 0$ and (3.B.10) implies that $\Pi_{\leq}\left(S_{j-1}u\Delta_j\nabla v\right) = 0$. Therefore in this case the two terms in (3.B.15) vanish. If $3\times 2^j \leq 2^{N-1}$ then $\Pi_{\leq}\Delta_j = \Delta_j$ and $\Pi_{\leq}\left(S_{j-1}u\Delta_j\nabla v\right) = S_{j-1}u\Delta_j\nabla v$, so the two terms in (3.B.15) are equal. Therefore there exists $C > 0$ independent from λ such that

$$\|A_1\|_{L^2} \leq \sum_{\frac{1}{C\lambda} \leq 2^j \leq \frac{C}{\lambda}} \left\| \left[S_{j-1}u, \Pi_{\leq}\right] \Delta_j \nabla v \right\|_{L^2}.$$

We now use (3.B.14) and (3.B.5) to obtain

$$\begin{aligned}
\|A_1\|_{L^2} &\lesssim \lambda \sum_{\frac{1}{C\lambda} \leq 2^j \leq \frac{C}{\lambda}} \|\nabla S_{j-1}u\|_{L^\infty} \|\Delta_j \nabla v\|_{L^2} \\
&\lesssim \lambda \|\nabla u\|_{L^\infty} \sum_{\frac{1}{C\lambda} \leq 2^j \leq \frac{C}{\lambda}} 2^j \|\Delta_j v\|_{L^2}
\end{aligned}$$

Using now the Cauchy-Schwarz inequality for finite sums, the characterization of L^2 given in Proposition 3.B.2 and $\left(\sum_{\frac{1}{C\lambda}\le 2^j\le \frac{C}{\lambda}} 2^{pj}\right)^{\frac{1}{p}} \lesssim \lambda^{-1}$ we obtain

$$\|A_1\|_{L^2} \lesssim \|\nabla u\|_{L^\infty} \|v\|_{L^2} . \tag{3.B.16}$$

For A_2, we have

$$[\Delta_j u, \Pi_\le] S_{j-1}\nabla v = \Delta_j u \Pi_\le (S_{j-1}\nabla v) - \Pi_\le (\Delta_j u S_{j-1}\nabla v) . \tag{3.B.17}$$

If $3\times 2^j \le 2^{N-1}$ then $\Pi_\le S_{j-1} = S_{j-1}$ and (3.B.10) still implies $\Pi_\le (\Delta_j u S_{j-1}\nabla v) = \Delta_j u S_{j-1}\nabla v$, so the two terms in (3.B.17) are equal. If $2^j \times \frac{1}{12} \ge 2^{N+1}$, then $\Pi_\le S_{j-1} = \Pi_\le$ and $\Pi_\le (\Delta_j u S_{j-1}\nabla v) = 0$. Therefore there exists C independent from λ such that

$$\|A_2\|_{L^2} \le \sum_{\frac{1}{C\lambda}\le 2^j\le \frac{C}{\lambda}} \|[\Delta_j u, \Pi_\le] S_{j-1}\nabla v\|_{L^2} + \sum_{\frac{C}{\lambda}\le 2^j} \|\Delta_j u \Pi_\le (\nabla v)\|_{L^2} .$$

We treat the first sum as we treated A_1:

$$\begin{aligned}\sum_{\frac{1}{C\lambda}\le 2^j\le \frac{C}{\lambda}} \|[\Delta_j u, \Pi_\le] S_{j-1}\nabla v\|_{L^2} &\lesssim \lambda \sum_{\frac{1}{C\lambda}\le 2^j\le \frac{C}{\lambda}} \|\nabla \Delta_j u\|_{L^\infty} \|S_{j-1}\nabla v\|_{L^2} \\ &\lesssim \|\nabla u\|_{L^\infty} \|v\|_{L^2} \lambda \sum_{\frac{1}{C\lambda}\le 2^j\le \frac{C}{\lambda}} 2^j \\ &\lesssim \|\nabla u\|_{L^\infty} \|v\|_{L^2} .\end{aligned}$$

For the second sum, we use (3.B.12) and then (3.B.6)

$$\begin{aligned}\sum_{\frac{C}{\lambda}\le 2^j} \|\Delta_j u \Pi_\le (\nabla v)\|_{L^2} &\lesssim \frac{1}{\lambda} \|v\|_{L^2} \sum_{\frac{C}{\lambda}\le 2^j} \|\Delta_j u\|_{L^\infty} \\ &\lesssim \frac{1}{\lambda} \|v\|_{L^2} \sum_{\frac{C}{\lambda}\le 2^j} 2^{-2j} 2^{\frac{7}{2}j} \|\Delta_j u\|_{L^2} \\ &\lesssim \lambda \|u\|_{H^{\frac{7}{2}}} \|v\|_{L^2}\end{aligned}$$

where we used the Cauchy-Schwarz inequality in ℓ^2, the characterization of $H^{\frac{7}{2}}$ given in Proposition 3.B.2 and $\left(\sum_{\frac{C}{\lambda}\le 2^j} 2^{-2j}\right)^{\frac{1}{2}} \lesssim \lambda^2$. We finally obtain

$$\|A_2\|_{L^2} \lesssim \left(\|\nabla u\|_{L^\infty} + \lambda \|u\|_{H^{\frac{7}{2}}}\right) \|v\|_{L^2} . \tag{3.B.18}$$

For A_3, we define

$$R_j = \sum_{i=j-1}^{j+1} [\Delta_i u, \Pi_\le] \Delta_j \nabla v = \sum_{i=j-1}^{j+1} \left(\Delta_i u \Pi_\le (\Delta_j \nabla v) - \Pi_\le (\Delta_i u \Delta_j \nabla v)\right) \tag{3.B.19}$$

so that $A_3 = \sum_j R_j$. If $2^j \times \frac{32}{3} \le 2^{N-1}$ then (3.B.11) implies $\Pi_\le (\Delta_i u \Delta_j \nabla v) = \Delta_i u \Delta_j \nabla v$ and (3.B.4) implies $\Pi_\le \Delta_j = \Delta_j$, so that the two terms in (3.B.19) are equal and $R_j = 0$. Therefore there exists a constant $C > 0$ independent from λ such that

$$\|A_3\|_{L^2} \le \sum_{\frac{C}{\lambda}\le 2^j} \sum_{i=j-1}^{j+1} \|[\Delta_i u, \Pi_\le] \Delta_j \nabla v\|_{L^2} .$$

We use (3.B.14), (3.B.5) and (3.B.6)

$$\begin{aligned}\|A_3\|_{L^2} &\lesssim \lambda \sum_{\frac{C}{\lambda}\leq 2^j} \sum_{i=j-1}^{j+1} 2^{\frac{5}{2}i+j} \|\Delta_i u\|_{L^2} \|\Delta_j v\|_{L^2} \\ &\lesssim \lambda \sum_{\frac{C}{\lambda}\leq 2^j} 2^{\frac{7}{2}j} \|\Delta_j u\|_{L^2} \|\Delta_j v\|_{L^2}\end{aligned}$$

where we neglect the sum over i. Using again the Cauchy-Schwarz inequality in ℓ^2, the characterization of $H^{\frac{7}{2}}$ and L^2 given in Proposition 3.B.2 we finally obtain

$$\|A_3\|_{L^2} \lesssim \lambda \|u\|_{H^{\frac{7}{2}}} \|v\|_{L^2} .$$

Together with (3.B.16) and (3.B.18) this last estimate concludes the proof of Lemma 3.2.2.

Chapter 4

Global existence for a toy model

4.1 Introduction

4.1.1 Presentation of the result

In this article we are interested in proving the global existence of *high-frequency solutions* to the following semi-linear wave system

$$\Box\Phi = Q(\partial\Phi, \partial\Phi) \tag{4.1.1}$$

on $\mathbb{R}_+ \times \mathbb{R}^3$ where $\Box = -\partial_t^2 + \Delta$ is the standard d'Alembertian operator and Q has the null structure (see Definition 4.2.1). The initial data we consider are highly oscillatory in the radial direction. This corresponds to a large data regime, which is to be compared with the well-known small data regime, see the discussion below. Let us give a rough version of the theorem proved in this article (see Theorem 4.3.1 for a precise version):

Theorem 4.1.1. *Consider sufficiently regular functions $(F_0, \varphi_0, \varphi_1)$ of size ε on $\mathbb{R}^3$ and let $\lambda > 0$. If ε is small enough (independent of λ), there exists a global solution Φ_λ to $\Box u = Q(\partial u, \partial u)$ of the form:*

$$\Phi_\lambda = \varphi + \lambda F \cos\left(\frac{t-r}{\lambda}\right) + \lambda^2 F_\lambda \tag{4.1.2}$$

where

- *φ satisfies $\Box\varphi = Q(\partial\varphi, \partial\varphi)$ and $(\varphi, \partial_t\varphi)_{|t=0} = (\varphi_0, \varphi_1)$,*
- *F satisfies $\left(\partial_t + \partial_r + \frac{1}{r}\right) F = F\partial\varphi$ and $F_{|t=0} = F_0$,*
- *F_λ satisfies $|F_\lambda| \lesssim \varepsilon$.*

In the next section we will review in detail our strategy of proof, for now let us just mention that the global existence of solutions to (4.1.1) on $\mathbb{R}^{3+1}$ with any (regular and small enough) initial data is a classical result of Klainerman (see [Kla86]), which introduced the concept of null quadratic forms precisely for this purpose, and Christodoulou (see [Chr86]) which used conformal compactification. Our proof relies crucially on the vector field method Klainerman introduced, but our theorem is *not* a consequence of his result for the following reasons:

- Theorem 4.1.1 can be summarized as follows: take some oscillating initial data containing terms like $\lambda\cos\left(\frac{r}{\lambda}\right)$, then the unique global solution arrising from those initial data actually presents the same oscillating behaviour. Unlike Klainerman, we not only prove global existence but propagate the shape of the high-frequency ansatz.

- Another crucial aspects of Theorem 4.1.1 is that the smallness threshold is independent of λ. One interesting aspects of high-frequency solutions of the form (4.1.2) is that their H^s norm is large for any $s > 1$, if λ is close to 0. Indeed, by differentiating (4.1.2) strictly more than once we obtain negative powers of λ. Therefore, if one wants to apply Klainerman's result with a fixed but small λ one needs to counterbalance the size of $\frac{1}{\lambda}$ with the smallness of the initial data (concretely by assuming something like $\varepsilon \ll \lambda$), see Remark 4.3.1. Therefore, our work gives an example of how one can relax the smallness assumption of Klainerman's result.

Our motivations for studying solutions of the form (4.1.2) to non-linear wave equations like (4.1.1) come from Burnett's conjecture in general relativity and are presented at the end of this introduction (see Section 4.1.3). However, highly oscillating solution to hyperbolic equations have been intensely studied in the field of *geometric optics*. In 1957, Lax laid the foundations of this field in his article [Lax57], where he proved that linear hyperbolic equations admit WKB solutions. Named after the physicists Wentzel, Kramers and Brillouin, WKB solutions were first introduced to understand the behaviour of quantum systems in a semi-classical regime, the main idea being to write some part of the solution as an asymptotic expansion in terms of the small parameter $\hbar$. In geometric optics, the small parameter is a wavelength.

In this article, we prove global existence of a highly oscillating solutions of a *non-linear* wave equation. The construction of high-frequency ansatz for non-linear systems has been first conducted by Choquet-Bruhat in [CB64], where she applies her result to deduce the ill-posedness of the Cauchy problem in C^1 for the systems she considers. In [JR92], Joly and Rauch extend the approximation procedure to higher space dimension. A complete overview of geometric optics can be found in the book of Rauch [Rau12].

In the limit $\lambda \to 0$, solutions of the form (4.1.2) explode in any $W^{k,p}$ if $k > 0$. Therefore, Theorem 4.1.1 allows us to consider large initial data for which we are still able to prove global existence. In [WY16], Wang and Yu also consider the large data regime by studying the *short pulse* ansatz, inspired by Christodoulou and his study of black holes formation.

4.1.2 Strategy of proof

Let us present our strategy of proof, which can be viewed as a high-frequency adaptation of Klainerman's vector field method.

4.1.2.1 Global existence for non-linear wave equations

In spatial dimension strictly larger than 3, the standard decay of waves is enough to ensure the global existence under the small initial data assumption of solutions to wave equations of the form

$$\Box u = (\partial u)^2. \tag{4.1.3}$$

However, since the counter example of John (see [Joh81]), it is known that solutions to (4.1.3) on $\mathbb{R}^{3+1}$ don't necessarily exist for all time, even in the small initial data regime. The globality of solutions depends on the actual structure of the quadratic non-linearity. In 1984, Klainerman introduced the famous *null condition* (see [Kla86] and Definition 4.2.1). The null condition can be summarized as follows: in every product $\partial u \partial u$ at least one of the derivatives is *tangent to the light cone*, and therefore enjoys better decay. This restriction on the non-linearity in (4.1.3) is sufficient (though not necessary) to obtain the existence of global solutions for small initial data.

In order to prove global existence of a solution, one powerful method is the vector field method, which goes back to Klainerman (see [Kla85]). Instead of using standard derivatives ∂_α to derive energy estimates, the heart of the method consists in using special vector fields linked to the symmetries of the Minkowski spacetime (see (4.2.3)). These vector fields are interesting for two reasons: they enjoy nice commutation properties with the wave operator and a more refined version of the classic Sobolev embedding $H^s \subset L^\infty$ for $s > \frac{3}{2}$ holds when replacing the standard derivatives in the H^s norm by the Minkowski vector fields (the so-called Klainerman-Sobolev inequality, see Proposition 4.2.2). This refined Sobolev embedding comes with decay rates in the null coordinates $t + r$ and $t - r$, which is the key argument to prove global existence.

This method can be improved by using a *weighted* energy estimate with an increasing function of $q = t - r$ as a weight. This weight implies the presence of an additional term on the LHS of the usual energy estimates (see Lemma 4.2.1), allowing to better control some of the derivatives of the solution (in our case the derivatives tangent to the light cone). This is known as the ghost weight method and was first introduced by Alinhac in [Ali01].

4.1.2.2 The high-frequency hierarchy

The main particularity of our result is the use of an high-frequency ansatz with a precise description of the oscillating terms, displaying what we call a *half-chessboard shape*. Let Φ_λ be the solution of (4.1.1) admitting the following formal high-frequency expansion

$$\Phi_\lambda(t,x) \sim \sum_{k\geq 0} \lambda^k \Phi^{(k)}\left(t, x, \frac{t-r}{\lambda}\right). \tag{4.1.4}$$

By half-chessboard shape, we mean that each $\Phi^{(i)}(t,x,\theta)$ contains several oscillating terms in the θ variable according to the following pattern:

$$\begin{aligned}
\Phi^{(0)} &= 1 \\
\Phi^{(1)} &= \cos(\theta) \\
\Phi^{(2)} &= \sin(\theta) + \cos(2\theta) \\
\Phi^{(3)} &= \cos(\theta) + \sin(2\theta) + \cos(3\theta) \\
&\vdots
\end{aligned} \tag{4.1.5}$$

Remark 4.1.1. *In (4.1.5), we only wrote down the oscillating terms, i.e the dependence of $\Phi^{(i)}$ on its third variable, and by $\Phi^{(0)} = 1$ we mean that it does not actually depends on the third variable, i.e it is not oscillating.*

The main challenge in defining a pattern of oscillation for each order is to capture the creation of harmonics due to the non-linear part of (4.1.1). It is not trivial that the half-chessboard shape is "stable" under quadratic non-linearity of the form $(\partial\Phi_\lambda)^2$, i.e that those terms only create harmonics already present in the ansatz for Φ_λ. This will be rigourously proved in Section 4.4, where we present the exact form of the ansatz for Φ_λ (see (4.4.1)). The non-linear interactions also create non-oscillating terms which can't be absorbed by the ansatz, unless we add to (4.1.5) some non-oscillating fields. To lighten this introduction we choose not to show them yet.

From a differential point of view, plugging oscillating terms $\lambda^k \cos\left(\frac{u}{\lambda}\right) F$ (with u a phase to be chosen later) in the wave operator gives

$$\Box\left(\lambda^k \cos\left(\frac{u}{\lambda}\right) F\right) = -\lambda^{k-2}\left(m^{\alpha\beta}\partial_\alpha u \partial_\beta u\right)\cos\left(\frac{u}{\lambda}\right) F - \lambda^{k-1}\sin\left(\frac{u}{\lambda}\right)\left(2m^{\alpha\beta}\partial_\alpha u\partial_\beta + \Box u\right) F + \lambda^k \cos\left(\frac{u}{\lambda}\right)\Box F \tag{4.1.6}$$

where $m^{\alpha\beta}\partial_\alpha u \partial_\beta u = -(\partial_t u)^2 + |\nabla u|^2$. We recover what is called the *geometric optics approximation*: in order to have $\Box\left(\lambda^k \cos\left(\frac{u}{\lambda}\right) F\right)$ at the same order as $\lambda^k \cos\left(\frac{u}{\lambda}\right) F$ we need

- $m^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$, i.e u to be a solution of the eikonal equation, or said differently we need the spacetime gradient of u, denoted ∂u, to be a null vector field for the Minkowski metric. In the rest of the article we will choose $u = t - r$.
- $\left(2m^{\alpha\beta}\partial_\alpha u\partial_\beta + \Box u\right) F = 0$, i.e F to be transported along the rays of u. If $u = t - r$, this rewrites

$$\left(\partial_t + \partial_r + \frac{1}{r}\right) F = 0.$$

The fact that u satisfies the eikonal equation has another consequence, related to our pattern of oscillation (4.1.5). Indeed since Q is a null form and ∂u is a null vector the following term vanishes

$$\left(\sin\left(\frac{u}{\lambda}\right) F^{(1)}\right)^2 Q(\partial u, \partial u) \tag{4.1.7}$$

where $F^{(1)}$ is the coefficient of $\Phi^{(1)}$. The presence of this term would totally break the half-chessboard shape since we would need $\Phi^{(1)}$ (as well as $\Phi^{(k)}$ for $k \geq 2$ because of the triangular structure) to contain all frequencies (see Remark 4.4.10).

The conclusion of this discussion is that solving (4.1.1) is equivalent to solving a *triangular* hierarchy of *linear* transport equations along the rays of u for each coefficients in the high-frequency expansion of Φ_λ, as well as a semi-linear wave equation for the remainder h, which is added to the formal expansion (4.1.4) to make it an exact solution of (4.1.1). Our ansatz is therefore of the following schematic form

$$\Phi_\lambda(t,x) = \sum_{k=0}^{K} \lambda^k \Phi^{(k)}\left(t, x, \frac{t-r}{\lambda}\right) + h_\lambda \tag{4.1.8}$$

where $h_\lambda = O\left(\lambda^K\right)$ solves an equation of the form

$$\Box h_\lambda = Q(\partial h_\lambda, \partial h_\lambda) + \lambda^K \Phi^{(\cdot)} \tag{4.1.9}$$

where Q is the null form and $\lambda^K \Phi^{(\cdot)}$ denotes terms of orders λ^K depending polynomially on the high-frequency waves $\Phi^{(k)}$ and their derivatives. Note that we deliberately forgot a lot of terms in (4.1.9), since from the point of view of the dependence in λ the quadratic terms in ∂h_λ are the most dangerous.

In (4.1.8), the degree of precision of the expansion defining Φ_λ is given by the integer K. Our work shows that we need $K \geq 4$ to close the bootstrap procedure for (4.1.9). This is due to the method used to prove global existence to (4.1.9), i.e the vector field method. Indeed, we want h_λ to satisfy bootstrap assumptions of the form $\left\| \partial Z^I h_\lambda \right\|_{L^2} \lesssim \lambda^{K-|I|}$ (where Z denotes any Minkowskian vector fields, see (4.2.3)). This mimics a high-frequency behaviour while h_λ is not oscillating. In order to improve these bootstrap assumptions, we will recover decay from the Klainerman-Sobolev inequality. This inequality allows one to bound $|f|$ by $\left\| Z^3 f \right\|_{L^2}$ with appropriate decay rates, which we omit in this introduction (see Proposition 4.2.2). The bootstrap assumptions for h_λ therefore imply that $|\partial h_\lambda| \lesssim \lambda^{K-4}$. This explains the lower bound on the degree of precision of the ansatz $K \geq 4$.

As a final remark on our strategy of proof, let us explain how we get decay for the high-frequency waves $\Phi^{(k)}$, which appear as source terms in (4.1.9). As explained above, the high-frequency waves solve a hierarchy of transport equation of the form

$$\left(\partial_t + \partial_r + \frac{1}{r} \right) f = g \tag{4.1.10}$$

with a RHS depending on the lower order high-frequency waves (thus respecting the triangular structure of the system). This equation rewrites $(\partial_t + \partial_r)(rf) = rg$, from which we get $|f| \lesssim r^{-1}$ if the RHS satisfies $|g| \lesssim r^{-3}$. See Proposition 4.5.2 for a precise statement. In the hierarchy of transport equations, the most dangerous term is $\Box \Phi^{(k)}$, i.e comes from the third line in (4.1.6). Schematically if we only consider this term we have

$$\left(\partial_t + \partial_r + \frac{1}{r} \right) \Phi^{(k+1)} = \Box \Phi^{(k)}.$$

Therefore, in order to apply Proposition 4.5.2 we need $\left| \Box \Phi^{(k)} \right| \lesssim r^{-3}$. We can't get this decay by commuting $\Box$ with the transport operator $\partial_t + \partial_r + \frac{1}{r}$ since it would only give $\left| \Box \Phi^{(k)} \right| \lesssim r^{-1}$. Instead we compute and estimate $\Box \Phi^{(k)}$ directly from $\partial_t + \partial_r + \frac{1}{r}$ in Lemma 4.A.2. See Remark 4.5.2 for a more detailled discussion of this procedure.

4.1.3 A toy model for general relativity

Here we present our principal motivation for studying the global existence of high-frequency solutions to (4.1.1).

4.1.3.1 The Einstein vacuum equations in wave coordinates

The Einstein equations are the main equations of the general relativity theory, they spell out the link between the curvature of the spacetime and the matter and energy it contains. On a Lorentzian manifold $(\mathcal{M}, g)$ and in the vacuum case they can be written in an elegant and compact form:

$$\mathrm{Ric}(g) = 0, \tag{4.1.11}$$

where $\mathrm{Ric}(g)$ is the Ricci tensor of the metric g. In order to define a initial value problem for this equation, we need to specify a coordinate system on $\mathcal{M}$. Noticed by Einstein in 1916 and used by Choquet-Bruhat to show local existence of the Einstein system in the 1950's, the *wave coordinates* define the most common coordinate system used in the study of the Einstein equations. In this coordinate system, the equation (4.1.11) becomes

$$\Box_g g_{\alpha\beta} = P_{\alpha\beta}(\partial g, \partial g), \tag{4.1.12}$$

where $\Box_g$ is the wave operator associated to g and $P_{\alpha\beta}$ is a quadratic non-linearity. Thus, the Einstein vacuum equations are recast as a quasi-linear hyperbolic system for the metric coefficients, which allows one to define and study a Cauchy problem (see [FB52] for the first proof of local well-posedness using wave coordinates).

One of the major question in general relativity is the question of the *stability* of particular solutions, such as the Minkowski spacetime, which, thanks to (4.1.12) reduces to a hyperbolic long-time existence problem. As it was proved by Choquet-Bruhat in [CB00], the non-linearity in (4.1.12) does not satisfy the null condition presented in Section 4.1.2. Note that the Einstein vacuum equations do satisfy a null condition through a dynamical gauge choice, see the seminal work [CK93]. However, the Einstein vacuum equations in wave gauge satisfy what Lindblad and Rodnianski call a *weak null condition*, allowing them to prove the non-linear stability of the Minkowski spacetime in this gauge in [LR10].

Therefore, studying the semi-linear equation $\Box u = Q(\partial u, \partial u)$ where Q is a null form is a toy model for the Einstein vacuum equations: it is a semi-linear equation instead of a quasi-linear equation, with the quadratic part being a null form instead of merely satisfying the weak null condition.

4.1.3.2 Burnett's conjecture

The Einstein vacuum equations are highly non-linear, which a priori allows *backreaction* to occur. This phenomenon can be described as follows: if a sequence of metrics $(g_\lambda)_\lambda$ converges in some sense to a background metric g_0, and if for all λ we have $\mathrm{Ric}(g_\lambda) = 0$, do we have $\mathrm{Ric}(g_0) = 0$? Said differently, can we pass to the limit in the Ricci tensor? Identifying the potential limit matter models occuring because of backreaction is called the Burnett conjecture, and was introduced by Burnett in [Bur89].

If the convergence $g_\lambda \to g_0$ happens in a very strong topology then the answer is obviously positive, but if the convergence is weak enough, we might obtain non-zero contributions from quadratic terms like $\partial(g_\lambda - g_0)\partial(g_\lambda - g_0)$. In the framework chosen by Burnett, the convergence is the following: g_λ converges strongly in L^∞ and ∂g_λ converges weakly in L^2. High-frequency ansatz of the form (4.1.4)-(4.1.5) precisely enjoy this type of convergence, and have been used to tackle Burnett's conjecture locally in time by Huneau and Luk in [HL18b]. This fully motivates the study of high-frequency solutions to (4.1.1).

Remark 4.1.2. *Even though the high-frequency solution Φ_λ converges in the Burnett sense to $\Phi^{(0)}$, no backreaction occurs in the case of the equation (4.1.1), meaning that the background limit solution $\Phi^{(0)}$ also satisfy (4.1.1) (see also (**BG**) in the reduced system). As Choquet-Bruhat noticed in [CB00], this is linked to the null condition satisfied by (4.1.1). Indeed, backreaction could only occur through the non-oscillating contributions of the squared sinus in (4.1.7), but as we saw (4.1.7) vanishes because Q is a null form.*

The absence of backreaction for (4.1.1) takes nothing away from the study of high-frequency solutions to this equation, as it is a first step towards proving the stability of high-frequency perturbation of some background metric g_0 close to Minkowski for the equation (4.1.12), which would require to incorporate the ideas of [LR10] to the high-frequency setting presented in this article.

4.2 Preliminaries

In this section, we describe our geometric and analytic setting, and introduce the key estimates used in this article.

4.2.1 Notations and function spaces

4.2.1.1 Coordinates and derivatives

We work in $\mathbb{R}_+ \times \mathbb{R}^3$ with the usual global coordinates system $(t = x_0, x_1, x_2, x_3)$. We also introduce the null coordinates

$$s = t + r \quad \text{and} \quad q = r - t.$$

For $t \geq 0$ we use the standard notation $\Sigma_t := \{t\} \times \mathbb{R}^3$. We also define the following subset of $\mathbb{R}_+ \times \mathbb{R}^3$ for some $R > 0$:

$$A^R := \left\{ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 \;\middle|\; R^{-1} \leq q \leq R \right\},$$
$$B^R := \left\{ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 \;\middle|\; q \leq R \right\},$$

and we set $A_0^R := A^R \cap \Sigma_0$. If $x = (x_1, x_2, x_3)$ and $r = |x|$, recall that $\partial_r = \sum_{i=1,2,3} \omega_i \partial_i$ where $\omega_i = \frac{x_i}{r}$. If $f : \mathbb{R}_+ \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a scalar function, then we define its spacetime gradient by the vector $\partial f = (\partial_t f, \partial_1 f, \partial_2 f, \partial_3 f)$. The "good" derivatives are defined to be

$$\overline{\partial}_0 = \frac{1}{2}(\partial_t + \partial_r) \quad \text{and} \quad \overline{\partial}_i = \partial_i - \frac{x_i}{r}\partial_r.$$

As we will see, they enjoy better decay than the other derivatives. In this article we consider $\mathbb{R}^d$-valued functions for some $d \geq 1$, i.e vectorial functions of the form $u = (u_1, \dots, u_d)$. For such a function we define

$$|\partial u| = \sum_{\substack{\alpha=0,\dots,3 \\ i=1,\dots,d}} |\partial_\alpha u_i| \quad \text{and} \quad |\overline{\partial} u| = \sum_{\substack{\alpha=0,\dots,3 \\ i=1,\dots,d}} |\overline{\partial}_\alpha u_i|.$$

4.2.1.2 Null forms

Now we define the class of quadratic non-linearties we consider in this article, i.e the classical null forms introduced by Klainerman.

Definition 4.2.1. *A quadratic form Q is said to be a null quadratic form (also called null form in the sequel) if it is a linear combination of Q_0 and $Q_{\alpha\beta}$ where*

$$Q_{\alpha\beta}(X,Y) = X_\alpha Y_\beta - X_\beta Y_\alpha \quad \textit{and} \quad Q_0(X,Y) = -X_0 Y_0 + \delta^{ij} X_i Y_j$$

for two vector fields X, Y on $\mathbb{R}_+ \times \mathbb{R}^3$, and $\alpha, \beta = 0,1,2,3$.

One can show that the null forms defined in this way are precisely the quadratic forms on the tangent bundle of Minkowski spacetime vanishing when contracted twice with the same null vector field.

Since the spacetime gradient of a scalar function is a vector field on $\mathbb{R}_+ \times \mathbb{R}^3$, this definition allows us to consider $Q(\partial u, \partial v)$ for u, v two *scalar* functions. However in this article, we would like to consider *vectorial* functions. Therefore if $u : \mathbb{R}_+ \times \mathbb{R}^3 \longrightarrow \mathbb{R}^d$ with $u = (u_1, \dots, u_d)$ and $v : \mathbb{R}_+ \times \mathbb{R}^3 \longrightarrow \mathbb{R}^{d'}$ with $v = (v_1, \dots, v_{d'})$, and if $Q_{k\ell}$ are null forms, we define

$$Q(\partial u, \partial v) := \sum_{\substack{k=1,\dots,d \\ \ell=1,\dots,d'}} Q_{k\ell}(\partial u_k, \partial v_\ell). \tag{4.2.1}$$

With this notation, we can write down properly the system studied in this article. For $d \geq 1$, we want to solve the system

$$\Box u_i = Q_i(\partial u, \partial u)$$

where $u : \mathbb{R}_+ \times \mathbb{R}^3 \longrightarrow \mathbb{R}^d$ is a vectorial function, i.e $u = (u_1, \ldots, u_d)$, and where Q_i is as in (4.2.1) for $i = 1, \ldots, d$. Though the Einstein equations motivates the study of a system of wave equations in this article, this plays no particular role here and from now on (with the exception of the statement of the main result) we will drop the indexes relative to $\mathbb{R}^d$ and simply use the condensed notation

$$\Box u = Q(\partial u, \partial u).$$

The null forms are in some sense the best quadratic non-linearity we can hope for in the sense that in each product $\partial u \partial u$, at least one of the derivatives is a good one, meaning that:

$$|Q(\partial u, \partial v)| \lesssim |\partial u||\overline{\partial} v| + |\overline{\partial} u||\partial v|. \tag{4.2.2}$$

4.2.1.3 Function spaces

In terms of function spaces, we use the usual L^p, C^m and Sobolev spaces defined on $\mathbb{R}^3$, together with weighted Sobolev spaces, defined by

Definition 4.2.2. *Let $m \in \mathbb{N}$, $1 < p < \infty$, $\delta \in \mathbb{R}$. The weighted Sobolev space $W^{m,p}_\delta$ is the completion of C^∞_0 under the norm*

$$\|u\|_{W^{m,p}_\delta} = \sum_{|\beta| \leq m} \left\| \langle x \rangle^{\delta + \beta} \nabla^\beta u \right\|_{L^p}.$$

We will use the notation $H^m_\delta = W^{m,2}_\delta$. The weighted Hölder space C^m_δ is the completion of C^m under the norm

$$\|u\|_{C^m_\delta} = \sum_{|\beta| \leq m} \left\| \langle x \rangle^{\delta + \beta} \nabla^\beta u \right\|_{L^\infty}.$$

Note that we defined scalar functions spaces. However by setting

$$\|u\|_X = \sum_{i=1,\ldots,d} \|u_i\|_X$$

for $u = (u_1, \ldots, u_d)$ and X any previously defined space, we extend those spaces and the corresponding norms to $\mathbb{R}^d$-valued functions. A weighted equivalent of the Sobolev embedding holds for those spaces (see [CB09] for a proof):

Proposition 4.2.1. *Let $s, m \in \mathbb{N}$. If $s \geq 2$ and $\beta \leq \delta + \frac{3}{2}$, then we have the continuous embedding*

$$H^{s+m}_\delta \hookrightarrow C^m_\beta.$$

4.2.2 Minkowski vector fields

We consider the set of Minkowski vector fields

$$\{\partial_\alpha, S = t\partial_t + r\partial_r, \Omega_{\alpha\beta} = x_\alpha\partial_\beta - x_\beta\partial_\alpha\} \tag{4.2.3}$$

and denote by Z^I any product of $|I|$ elements of this set, where I is an 11-dimensional integer. Note that since $x_0 = -x^0$, we have $\Omega_{0i} = -t\partial_i - x_i\partial_t$. These vector fields allow us to recover the usual derivatives ∂ while gaining a weight, whose nature explain why the derivatives $\overline{\partial}$ are called "good" derivatives:

$$|\partial u| \lesssim \frac{1}{1+|q|}\sum_{|I|=1}|Z^I u| \quad \text{and} \quad |\overline{\partial} u| \lesssim \frac{1}{1+s}\sum_{|I|=1}|Z^I u|. \tag{4.2.4}$$

The Minkowski vector fields enjoy good commutativity properties with the wave operator:

$$[\Box, Z] = 0$$

except for S, which satisfies $[\Box, S] = 2\Box$. This implies that

$$|\Box Z^I u| \lesssim \sum_{|I'|\leq|I|}|Z^{I'}\Box u|. \tag{4.2.5}$$

The Minkowski vector fields behave nicely with the null forms, in the sense that the following holds:

$$|Z^I Q(\partial u, \partial v)| \lesssim \sum_{|I_1|+|I_2|\leq|I|}|Q(\partial Z^{I_1}u, \partial Z^{I_2}v)| \tag{4.2.6}$$

4.2.3 Useful estimates

In this section we collect different estimates crucially used in this paper.

4.2.3.1 Weighted Klainerman-Sobolev inequality

We start by an estimate which allows us to bound the L^∞ norm of a function knowing the L^2 norm of some of its derivatives with weights.

Proposition 4.2.2. *Consider the weight*

$$w(q) := \begin{cases} 1 + (1+|q|)^{-\alpha} & \text{for} \quad q < 0 \\ 1 + (1+|q|)^{\beta} & \text{for} \quad q > 0 \end{cases}$$

with $1 < \beta < 3$ and $\alpha > 0$. Let f be compactly supported, we have

$$|w^{\frac{1}{2}}(q) f| \lesssim \frac{1}{(1+s)\sqrt{1+|q|}}\sum_{|I|\leq 3}\left\|w^{\frac{1}{2}}(q) Z^I f\right\|_{L^2}. \tag{4.2.7}$$

This is the so-called weighted Klainerman-Sobolev inequality, for which a proof can be found in [LR10]. The assumption on f in the previous proposition can be relaxed: the proposition holds as long as the RHS of (4.2.7) is finite.

4.2.3.2 Weighted energy estimate

As is standard, our strategy of proof relies on energy estimates for the wave operator $\Box$.

Lemma 4.2.1. *Let $w(q)$ be a non-decreasing weight function. Then*

$$\int_{\Sigma_t} |\partial\varphi|^2 w(q)\mathrm{d}x + \int_0^t\int_{\Sigma_\tau} |\overline{\partial}\varphi|^2 w'(q)\mathrm{d}x\mathrm{d}\tau \lesssim \int_{\Sigma_0} |\partial\varphi|^2 w(q)\mathrm{d}x + \int_0^t\int_{\Sigma_\tau} |\Box\varphi\partial_t\varphi| w(q)\mathrm{d}x\mathrm{d}\tau.$$

This lemma is the semi-linear equivalent of Lemma 6.1 in [LR10]. It gives a *weighted* energy estimate, with a weight depending only on q. This is called the *ghost weight* method and was first introduced by Alinhac in [Ali01]. Depending on the choice of the weight, it gives us extra decay for solutions of a wave equation in the region $q > 0$, i.e the exterior of the light-cone. The other advantage of this method is the presence of a space-time integral on the LHS involving only the "good" derivatives of the solution.

The usual energy estimate for the wave operator is obtained when $w(q) \equiv 1$, and one of the standard application of this case is the *finite speed of propagation* for solutions of semi-linear wave equation.

Lemma 4.2.2. *We define*

$$C_{t_0,x_0} = \left\{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 \;\middle|\; 0 \leq t \leq t_0 \quad \text{and} \quad |x - x_0| \leq t_0 - t\right\}$$

and $B(x_0,t_0)$ is the ball in $\mathbb{R}^3$ centered at x_0 of radius t_0. Let $P : \mathbb{R}^{3+1} \to \mathbb{R}$ a smooth function such that

$$|P(X)| \lesssim |X| + |X|^2.$$

Let $(t_0,x_0) \in \mathbb{R}_+ \times \mathbb{R}^3$ and $u : \mathbb{R}^{3+1} \to \mathbb{R}$ a solution of $\Box u = P(\partial u)$ in C_{t_0,x_0}. If $(u,\partial_t u)$ initially vanishes on $B(x_0,t_0)$, then $u_{|C_{t_0,x_0}} = 0$.

This lemma will be used to localize the different terms in the decomposition of our solution (see (4.4.1)).

4.3 Main result

We now state the main result of this article.

Theorem 4.3.1. *Let $d \geq 1$, $N \geq 26$, $R > 0$, $-\frac{1}{2} < \delta < \frac{1}{2}$, $F_0 \in C^{N-4}(\mathbb{R}^3,\mathbb{R}^d)$ supported in A_0^R and $(\varphi_0,\varphi_1) \in H_\delta^{N+1}(\mathbb{R}^3,\mathbb{R}^d) \times H_{\delta+1}^N(\mathbb{R}^3,\mathbb{R}^d)$ such that*

$$\|F_0\|_{C^{N-4}} + \|\varphi_0\|_{H_\delta^{N+1}} + \|\varphi_1\|_{H_{\delta+1}^N} \leq \varepsilon.$$

There exists $\varepsilon_0 = \varepsilon_0(N,R,\delta) > 0$ such that if $\varepsilon \leq \varepsilon_0$ and $\lambda \in (0,1]$ there exists a unique global solution $\Phi_\lambda : \mathbb{R}_+ \times \mathbb{R}^3 \longrightarrow \mathbb{R}^d$ to

$$\begin{cases} \Box\Phi_{\lambda,i} &= Q_i(\partial\Phi_\lambda,\partial\Phi_\lambda) \\ \Phi_{\lambda|t=0} &= \varphi_0 + \lambda F_0\cos\left(\frac{r}{\lambda}\right) \\ \partial_t\Phi_{\lambda|t=0} &= \varphi_1 + F_0\sin\left(\frac{r}{\lambda}\right) + \lambda\tilde{F}_{0,\lambda} \end{cases}$$

where Q_i is a null form and $\tilde{F}_{0,\lambda}$ is supported in A_0^R depending only on $(\varphi_0,\varphi_1,F_0)$ and which satisfies $\left\|\tilde{F}_{0,\lambda}\right\|_{L^\infty} \lesssim \varepsilon$. The function Φ_λ admits the decomposition

$$\Phi_\lambda = \varphi + \lambda F\cos\left(\frac{t-r}{\lambda}\right) + \lambda^2\tilde{F}_\lambda,$$

where

- *$\varphi \in H^{N+1}(\mathbb{R}_+ \times \mathbb{R}^3, \mathbb{R}^d)$ satisfies $\Box \varphi_i = Q_i(\partial\varphi, \partial\varphi)$ and $(\varphi, \partial_t \varphi)_{|t=0} = (\varphi_0, \varphi_1)$,*

- *$F \in C^{N-4}(\mathbb{R}_+ \times \mathbb{R}^3, \mathbb{R}^d)$ is supported in A^R and satisfies*

$$\left(\partial_t + \partial_r + \frac{1}{r}\right) F_i = \frac{1}{2} \sum_{k,\ell=1,\ldots,d} F_k Q_i(\partial(t-r), \partial\varphi_\ell)$$

  and $F_{|t=0} = F_0$,

- *$\tilde{F}_\lambda$ is supported in B^R and it is such that $|\tilde{F}_\lambda| \lesssim \varepsilon$ and $|\partial \tilde{F}_\lambda| \lesssim \frac{\varepsilon}{\lambda}$.*

Remark 4.3.1. *The main feature of this theorem is that the smallness constant ε_0 does not depend on λ, which is allowed to take any value in $(0,1]$. In addition to the nature of the decomposition of Φ_λ, this is at the heart of the high-frequency nature of the solution we construct: if we let λ tends to 0, the high order norms of Φ_λ (precisely the $W^{\infty,p}$ norm for $p > 1$) blows up but this is not compensated by letting ε_0 go to 0 as well. Therefore, the limit of Φ_λ as λ tends to 0 is meaningful, and we have:*

$$\Phi_\lambda \to \varphi, \quad \text{uniformly in } L^\infty,$$
$$\partial\Phi_\lambda \rightharpoonup \partial\varphi, \quad \text{weakly in } L^2.$$

4.4 Construction of the ansatz

The heart of the proof of Theorem 4.3.1 is to construct a high-frequency ansatz for the solution Φ_λ. The main purpose of this ansatz is to capture the creation of higher-order harmonics through the quadratic non-linearity. Those harmonics may also include non-oscillating terms. As explained in the introduction, we choose an ansatz with a *half-chessboard* shape. More precisely we consider a solution Φ_λ of the form

$$\Phi_\lambda = \varphi + \sum_{2 \leq k \leq K-1} \lambda^k \psi^{(k)} + \sum_{1 \leq i \leq k \leq K} \lambda^k F^{(k,i)} \mathbf{T}_\lambda^{(k,i)} + h_\lambda, \tag{4.4.1}$$

where $K \geq 2$ is an integer and where we use the following notation for and $k, i \in \mathbb{Z}$:

$$\mathbf{T}_\lambda^{(k,i)} = \begin{cases} \sin\left(\frac{i(t-r)}{\lambda}\right) & \text{if } k+i \text{ is odd}, \\ \cos\left(\frac{i(t-r)}{\lambda}\right) & \text{if } k+i \text{ is even}. \end{cases} \tag{4.4.2}$$

Besides from being at the very heart of what we call an half-chessboard shaped ansatz, this notation is useful to capture the behaviours of the standard trigonometric functions regarding products and derivation (see Lemma 4.4.1). A bit of vocabulary:

- φ is called the *background* solution,

- the functions $\psi^{(\ell)}$ are called the *non-oscillating* terms of the ansatz, whereas $F^{(k,i)}\mathbf{T}_\lambda^{(k,i)}$ are the *oscillating* terms,

- the function h_λ is the remainder of the ansatz and is non-oscillating, however it mimics a high-frequency behaviour, i.e it loses one power of λ for each derivatives, a fact clearly seen on the schematic estimates it will satisfy:

$$\|\partial^m h_\lambda\|_{L^2} \lesssim \lambda^{K+1-m} \quad \text{for} \quad m \geq 1.$$

The degree of precision of the ansatz (4.4.1) is given by the integer K. Our work is valid for all values of $K \geq 2$, but shows that global existence as a solution of (4.1.1) requires $K \geq 4$. As explained in the introduction, this is due to the Klainerman-Sobolev inequality.

Remark 4.4.1. *Since we adopt the standard convention that sums over empty sets vanish, there are no $\psi^{(\ell)}$ functions in the ansatz if $K = 2$. However, one major interest of our result is precisely the freedom we have on K: even though we are interested in a lower bound on K to ensure the global existence of Φ_λ, our proof also shows that we can describe the propagation of high-frequency waves with an arbitrary large precision.*

Remark 4.4.2. *In the rest of the article we will use the following convention: $\psi^{(1)} = 0$ and $F^{(k,i)} = 0$ if $i > k$ or $k = 0$. This will be useful in the sequel not to be too bothered with complicated sums.*

Our strategy of proof is to derive formally from $\Box\Phi_\lambda = Q(\partial\Phi_\lambda, \partial\Phi_\lambda)$ a system of equations for the unknowns $(\varphi, \psi^{(\ell)}, F^{(k,i)}, h_\lambda)$ and prove global existence for this system, called in the rest of this article the *reduced system*. Let us start by stating a technical lemma summarizing some basic properties of the trigonometric functions $\mathbf{T}_\lambda^{(k,i)}$ regarding products, the wave operator and the null forms.

Remark 4.4.3. *In the next lemma and in the rest of the article (including the technical next section), we won't write the numerical constants which appear in the computations, meaning for example that an equality like (4.4.3) has to be understood in the following sense: there exists real constants $A = A(k,\ell,i,j)$ and $B = B(k,\ell,i,j)$ such that*

$$\mathbf{T}_\lambda^{(k,i)}\mathbf{T}_\lambda^{(\ell,j)} = A\mathbf{T}_\lambda^{(k+\ell,i+j)} + B\mathbf{T}_\lambda^{(k-\ell,i-j)}.$$

Though these numerical constants can depend on every indexes appearing in (4.4.1), they are not allowed to depend on λ, meaning that we will always keep trace of λ in the formal computations.

Lemma 4.4.1. *Let $k, \ell, i, j \in \mathbb{Z}$, f and g any function and Q any null form. We have $\mathbf{T}_\lambda^{(k,i)} = \mathbf{T}_\lambda^{(|k|,|i|)}$ and*

$$\mathbf{T}_\lambda^{(k,i)}\mathbf{T}_\lambda^{(\ell,j)} = \mathbf{T}_\lambda^{(k+\ell,i+j)} + \mathbf{T}_\lambda^{(k+\ell,i-j)}, \tag{4.4.3}$$

$$\Box\left(g\mathbf{T}_\lambda^{(k,i)}\right) = \Box g\mathbf{T}_\lambda^{(k,i)} + \frac{1}{\lambda}\mathscr{L}g\mathbf{T}_\lambda^{(k-1,i)}, \tag{4.4.4}$$

$$Q\left(\partial\left(f\mathbf{T}_\lambda^{(k,i)}\right), \partial g\right) = Q(\partial f, \partial g)\mathbf{T}_\lambda^{(k,i)} + \frac{1}{\lambda}f\omega\overline{\partial}g\mathbf{T}_\lambda^{(k-1,i)}, \tag{4.4.5}$$

$$\begin{aligned} Q\left(\partial\left(f\mathbf{T}_\lambda^{(k,i)}\right), \partial\left(g\mathbf{T}_\lambda^{(\ell,j)}\right)\right) &= Q(\partial f, \partial g)\mathbf{T}_\lambda^{(k,i)}\mathbf{T}_\lambda^{(\ell,j)} \\ &\quad + \frac{1}{\lambda}f\omega\overline{\partial}g\mathbf{T}_\lambda^{(k-1,i)}\mathbf{T}_\lambda^{(\ell,j)} + \frac{1}{\lambda}g\omega\overline{\partial}f\mathbf{T}_\lambda^{(k,i)}\mathbf{T}_\lambda^{(\ell-1,j)}, \end{aligned} \tag{4.4.6}$$

where $\mathscr{L}$ is the following transport operator

$$\mathscr{L} := \partial_t + \partial_r + \frac{1}{r}$$

and where $\omega\overline{\partial}$ stands for a linear combination of $\omega_i\overline{\partial}_\alpha$ for $i = 1,2,3$ and $\alpha = 0,1,2,3$.

Proof. The proof of (4.4.3) is a direct rewriting with our notation (4.4.2) of the usual linearization formulas

$$\begin{aligned} 2\cos(a)\cos(b) &= \cos(a-b) + \cos(a+b), \\ 2\sin(a)\cos(b) &= \sin(a-b) + \sin(a+b), \\ 2\sin(a)\sin(b) &= \cos(a-b) - \cos(a+b). \end{aligned}$$

The proof of (4.4.4) follows from a direct computation, and even though $\Box$ is a second order operator, we don't lose two powers of λ since the phase $t-r$ satisfies the eikonal equation for the Minkowski metric, i.e

$$m^{\alpha\beta}\partial_\alpha(t-r)\partial_\beta(t-r) = 0. \tag{4.4.7}$$

The proof of (4.4.5) follows from the computations

$$\begin{aligned} Q_0(\partial(t-r), \partial h) &= -2\overline{\partial}_0 h, \\ Q_{0i}(\partial(t-r), \partial h) &= \overline{\partial}_i h + 2\omega_i \overline{\partial}_0 h, \\ Q_{ij}(\partial(t-r), \partial h) &= \omega_j \overline{\partial}_i h - \omega_i \overline{\partial}_j h, \end{aligned}$$

(which holds for any scalar function h) and the definition of a null form as in Definition 4.2.1.

The formula (4.4.6) follows directly from (4.4.5). □

Remark 4.4.4. *Note that even though a null form only involves first order derivatives, we would expect some $\frac{1}{\lambda^2}$ terms in (4.4.6) since $Q(\partial u, \partial v)$ is a quadratic expression. It is not the case since null forms are precisely the quadratic forms vanishing on null vector fields in Minkowski space, and (4.4.7) precisely means that the space-time gradient of $t-r$ is a null vector field.*

Remark 4.4.5. *Since $|\partial^{|\alpha|}\omega_i| \lesssim 1$ in the region $\{r \geq \frac{1}{R}\}$ for any multi-index α and since formulas (4.4.5) and (4.4.6) will be applied with f supported in A^R, the ω factors in (4.4.5) and (4.4.6) are irrelevant in our estimates and we omit them in the sequel.*

4.4.1 High-frequency expansion of the semi-linear wave equation

In this section, we plug the ansatz (4.4.1) into the equation

$$\Box\Phi_\lambda = Q(\partial\Phi_\lambda, \partial\Phi_\lambda). \tag{4.4.8}$$

The computations are rather tedious, and we present them in the form of two lemmas, one for the LHS of (4.4.8) and one for its RHS.

4.4.1.1 Expansion of the wave operator

We start with the RHS of (4.4.8).

Lemma 4.4.2. *If Φ_λ is defined by (4.4.1) we have*

$$\Box\Phi_\lambda = \sum_{0\leq p\leq K} \lambda^p \left(\Box\Phi_\lambda\right)^{(p)}$$

with

$$\left(\Box\Phi_\lambda\right)^{(0)} = \Box\varphi + \mathscr{L}F^{(1,1)}\mathbf{T}_\lambda^{(0,1)} \tag{4.4.9}$$

$$\left(\Box\Phi_\lambda\right)^{(K)} = \sum_{1\leq i\leq K} \Box F^{(K,i)}\mathbf{T}_\lambda^{(K,i)} + \frac{1}{\lambda^K}\Box h_\lambda \tag{4.4.10}$$

and

$$(\Box\Phi_\lambda)^{(p)} = \sum_{1\leq i\leq p+1}\left(\mathscr{L}F^{(p+1,i)} + \Box F^{(p,i)}\right)\mathbf{T}_\lambda^{(p,i)} + \Box\psi^{(p)} \tag{4.4.11}$$

for $1 \leq p \leq K-1$.

The proof of this lemma is left to the reader and is an application of (4.4.4) together with the convention stated in Remark 4.4.2.

4.4.1.2 Expansion of the null form

Lemma 4.4.3. *If* Φ_λ *is defined by* (4.4.1) *we have*

$$Q(\partial\Phi_\lambda, \partial\Phi_\lambda) = \sum_{0\leq k\leq K-1}\lambda^k Q^{(k)} + \lambda^K Q^{(\geq K)}$$

with

$$Q^{(0)} = Q(\partial\varphi, \partial\varphi) + F^{(1,1)}\overline{\partial}\varphi\mathbf{T}_\lambda^{(0,1)}, \tag{4.4.12}$$

$$\begin{aligned} Q^{(\geq K)} &= \frac{1}{\lambda^K}\mathcal{N}(h_\lambda) + Q(\partial F, \partial\varphi)\mathbf{T}_\lambda \\ &\quad + \lambda^+\left(F\overline{\partial}F\mathbf{T}_\lambda + Q(\partial F, \partial F)\mathbf{T}_\lambda + Q(\partial F, \partial\psi)\mathbf{T}_\lambda + F\overline{\partial}\psi\mathbf{T}_\lambda + Q(\partial\psi, \partial\psi)\right), \end{aligned} \tag{4.4.13}$$

and

$$\begin{aligned} &Q^{(p)} \\ &= Q(\partial\psi^{(p)}, \partial\varphi) + \sum_{2\leq k\leq p-2} Q(\partial\psi^{(k)}, \partial\psi^{(p-k)}) \\ &+ \sum_{1\leq i\leq p+1}\left(F^{(p+1,i)}\overline{\partial}\varphi + Q(\partial F^{(p,i)}, \partial\varphi)\right)\mathbf{T}_\lambda^{(p,i)} \\ &+ \sum_{\substack{1\leq i\leq k\leq p \\ 1\leq j\leq p-k+1}}\left(Q(\partial F^{(k,i)}, \partial F^{(p-k,j)}) + F^{(k,i)}\overline{\partial}F^{(p-k+1,j)} + F^{(p-k+1,j)}\overline{\partial}F^{(k,i)}\right)\left(\mathbf{T}_\lambda^{(p,i+j)} + \mathbf{T}_\lambda^{(p,i-j)}\right) \\ &+ \sum_{\substack{0\leq k\leq p-1 \\ 1\leq i\leq k+1}}\mathbf{T}_\lambda^{(k,i)}\left(F^{(k+1,i)}\overline{\partial}\psi^{(p-k)} + Q(\partial F^{(k,i)}, \partial\psi^{(p-k)})\right) \end{aligned} \tag{4.4.14}$$

for $1 \leq p \leq K-1$, *and where* $\mathcal{N}(h_\lambda)$ *is defined in* (4.4.16).

Remark 4.4.6. *In this lemma, we use the following notations for clarity:*

- F *or* ψ *denotes linear combinations of terms of the families* $F^{(k,i)}$ *or* $\psi^{(\ell)}$,
- $\mathbf{T}_\lambda$ *denotes linear combinations of terms of the family* $\mathbf{T}_\lambda^{(k,i)}$,
- *in front of a linear combination, the symbol* λ^+ *means that the coefficients of the linear combination contains some positive powers of* λ.

Proof. Using (4.4.5)-(4.4.6), we obtain by direct expansion

$$\begin{aligned}
Q(\partial\Phi_\lambda,\partial\Phi_\lambda) &= Q(\partial\varphi,\partial\varphi) + Q(\partial h_\lambda,\partial h_\lambda) + Q(\partial h_\lambda,\partial\varphi) + \lambda^+ Q(\partial(F\mathbf{T}_\lambda),\partial h_\lambda) + \lambda^+ Q(\partial\psi,\partial h_\lambda)\\
&+ \sum_{1\leq i\leq k\leq K} \lambda^k Q(\partial F^{(k,i)},\partial\varphi)\mathbf{T}_\lambda^{(k,i)} + \sum_{1\leq i\leq k\leq K} \lambda^{k-1} F^{(k,i)}\overline{\partial}\varphi\mathbf{T}_\lambda^{(k-1,i)}\\
&+ \sum_{2\leq k\leq K-1} \lambda^k Q(\partial\psi^{(k)},\partial\varphi)\\
&+ \sum_{\substack{1\leq k,\ell\leq K\\ 1\leq i\leq k\\ 1\leq j\leq \ell}} \lambda^{k+\ell-1}\left(F^{(k,i)}\overline{\partial}F^{(\ell,j)} + F^{(\ell,j)}\overline{\partial}F^{(k,i)}\right)\left(\mathbf{T}_\lambda^{(k+\ell-1,i+j)} + \mathbf{T}_\lambda^{(k+\ell-1,i-j)}\right)\\
&+ \sum_{\substack{1\leq k,\ell\leq K\\ 1\leq i\leq k\\ 1\leq j\leq \ell}} \lambda^{k+\ell} Q(\partial F^{(k,i)},\partial F^{(\ell,j)})\left(\mathbf{T}_\lambda^{(k+\ell,i+j)} + \mathbf{T}_\lambda^{(k+\ell,i-j)}\right)\\
&+ \sum_{\substack{1\leq i\leq k\leq K\\ 2\leq \ell\leq K-1}} \lambda^{k+\ell} Q(\partial F^{(k,i)},\partial\psi^{(\ell)})\mathbf{T}_\lambda^{(k,i)} + \sum_{\substack{1\leq i\leq k\leq K\\ 2\leq \ell\leq K-1}} \lambda^{k+\ell-1} F^{(k,i)}\overline{\partial}\psi^{(\ell)}\mathbf{T}_\lambda^{(k-1,i)}\\
&+ \sum_{2\leq k,\ell\leq K-1} \lambda^{k+\ell} Q(\partial\psi^{(k)},\partial\psi^{(\ell)})
\end{aligned} \tag{4.4.15}$$

Note that in order to simplify the terms $F\overline{\partial}F$ we used (4.4.3) and its consequence

$$\mathbf{T}_\lambda^{(k-1,i)}\mathbf{T}_\lambda^{(\ell,j)} = \mathbf{T}_\lambda^{(k,i)}\mathbf{T}_\lambda^{(\ell-1,j)} = \mathbf{T}_\lambda^{(k+\ell-1,i+j)} + \mathbf{T}_\lambda^{(k+\ell-1,i-j)}.$$

Now we want to regroup terms by their λ powers. For this we use the following facts

$$\begin{aligned}
\sum_{1\leq k,\ell\leq K} \lambda^{k+\ell} a_{k,\ell} &= \sum_{2\leq p\leq K-1} \lambda^p \sum_{1\leq k\leq p-1} a_{k,p-k} + O\left(\lambda^K\right)\\
\sum_{\substack{1\leq k\leq K\\ 2\leq \ell\leq K-1}} \lambda^{k+\ell} a_{k,\ell} &= \sum_{3\leq p\leq K-1} \lambda^p \sum_{1\leq k\leq p-2} a_{k,p-k} + O\left(\lambda^K\right)\\
\sum_{2\leq k,\ell\leq K-1} \lambda^{k+\ell} a_{k,\ell} &= \sum_{4\leq p\leq K-1} \lambda^p \sum_{2\leq k\leq p-2} a_{k,p-k} + O\left(\lambda^K\right)
\end{aligned}$$

which holds for every families $(a_{k,\ell})_{(k,\ell)\in\mathbb{N}^2}$. Therefore if $1\leq p\leq K-1$ we have

$$\begin{aligned}
Q^{(p)} = \sum_{1\leq i\leq p} & Q(\partial F^{(p,i)},\partial\varphi)\mathbf{T}_\lambda^{(p,i)} + \sum_{1\leq i\leq p+1} F^{(p+1,i)}\overline{\partial}\varphi\mathbf{T}_\lambda^{(p,i)} + Q(\partial\psi^{(p)},\partial\varphi)\\
&+ \sum_{\substack{1\leq i\leq k\leq p-1\\ 1\leq j\leq p-k}} Q(\partial F^{(k,i)},\partial F^{(p-k,j)})\left(\mathbf{T}_\lambda^{(p,i+j)} + \mathbf{T}_\lambda^{(p,i-j)}\right)\\
&+ \sum_{1\leq i\leq k\leq p-1} Q(\partial F^{(k,i)},\partial\psi^{(p-k)})\mathbf{T}_\lambda^{(k,i)} + \sum_{2\leq k\leq p-2} Q(\partial\psi^{(k)},\partial\psi^{(p-k)})\\
&+ \sum_{\substack{1\leq i\leq k\leq p\\ 1\leq j\leq p-k+1}} \left(F^{(k,i)}\overline{\partial}F^{(p-k+1,j)} + F^{(p-k+1,j)}\overline{\partial}F^{(k,i)}\right)\left(\mathbf{T}_\lambda^{(p,i+j)} + \mathbf{T}_\lambda^{(p,i-j)}\right)\\
&+ \sum_{1\leq i\leq k\leq p-1} F^{(k,i)}\overline{\partial}\psi^{(p-k+1)}\mathbf{T}_\lambda^{(k-1,i)}
\end{aligned}$$

where we used several times our convention on $\psi^{(1)}$ and on sums over empty sets. If we regroup terms according to their oscillating behaviour in this expression we obtain (4.4.14).

It remains to compute $Q^{(0)}$ and $Q^{(\geq K)}$. For $Q^{(0)}$, we simply look at (4.4.15). For $Q^{(\geq K)}$, we don't need to be precise in terms of the oscillating behaviour and the hierarchy since $\Box h_\lambda$ will absorb everything. This explains the expression of the lemma, with the following definition:

$$\mathcal{N}(h_\lambda) = Q(\partial h_\lambda, \partial h_\lambda) + Q(\partial h_\lambda, \partial\varphi) + \lambda^+ Q(\partial(F\mathbf{T}_\lambda), \partial h_\lambda) + \lambda^+ Q(\partial\psi, \partial h_\lambda). \tag{4.4.16}$$

$\square$

4.4.2 The reduced system

As the two previous lemmas show, solving

$$\Box\Phi_\lambda = Q(\partial\Phi_\lambda, \partial\Phi_\lambda)$$

is equivalent to solving the following high-frequency hierarchy:

$$(\Box\Phi_\lambda)^{(p)} = Q^{(p)} \tag{4.4.17}$$

for $0 \leq p \leq K-1$ and

$$(\Box\Phi_\lambda)^{(K)} = Q^{(\geq K)}. \tag{4.4.18}$$

Since h_λ is not-oscillating, (4.4.18) simply rewrites as a semi-linear wave equation with oscillating coefficients:

$$\begin{aligned}\Box h_\lambda = {} & \mathcal{N}(h_\lambda) + \lambda^K Q(\partial F, \partial\varphi)\mathbf{T}_\lambda + \lambda^K \Box F \mathbf{T}_\lambda \\ & + \lambda^K \lambda^+ \left(F\overline{\partial} F\mathbf{T}_\lambda + Q(\partial F, \partial F)\mathbf{T}_\lambda + Q(\partial F, \partial\psi)\mathbf{T}_\lambda + F\overline{\partial}\psi\mathbf{T}_\lambda + Q(\partial\psi, \partial\psi) \right)\end{aligned}$$

For the remaining equations of order λ^p for $0 \leq p \leq K-1$, note that both the LHS and RHS contain non-oscillating and oscillating terms, which we have to identify precisely.

For $p = 0$ it is quite straightforward: the non-oscillating part can be absorbed by the background equation

$$\Box\varphi = Q(\partial\varphi, \partial\varphi)$$

and the part oscillating like $\mathbf{T}_\lambda^{(0,1)}$ is absorbed thanks to a transport equation for $F^{(1,1)}$:

$$\mathscr{L} F^{(1,1)} = F^{(1,1)}\overline{\partial}\varphi.$$

If $1 \leq p \leq K-1$, the distribution between oscillating and non-oscillating terms is less clear, especially since the interaction between the different $F^{(k,i)}$ leads to the creation of non-oscillating terms. This is a consequence of the presence of $\mathbf{T}_\lambda^{(p,i-j)}$ in (4.4.14), which is non-oscillating if and only if p is even and $i = j$. These created non-oscillating terms are the motivation behind the terms $\psi^{(\ell)}$, indeed they are absorbed by $\Box\psi^{(\ell)}$ present in (4.4.11). Therefore, since p has to be even so that $\mathbf{T}_\lambda^{(p,i-j)}$ is non-oscillating, we only need $\psi^{(\ell)}$ for ℓ even and we take $\psi^{(\ell)} = 0$ for ℓ odd.

This has the following nice consequence: in the last term of $Q^{(p)}$ (for $1 \leq p \leq K-1$)

$$\sum_{\substack{0\leq k\leq p-1 \\ 1\leq i\leq k+1}} \mathbf{T}_\lambda^{(k,i)} \left(F^{(k+1,i)}\overline{\partial}\psi^{(p-k)} + Q(\partial F^{(k,i)}, \partial\psi^{(p-k)}) \right)$$

we now impose that $p-k$ is even, which implies $\mathbf{T}_\lambda^{(k,i)} = \mathbf{T}_\lambda^{(p,i)}$ (since p and k have the same parity). Therefore, the oscillating terms in $Q^{(p)}$ are all of the form $\mathbf{T}_\lambda^{(p,i)}$ for $1 \leq i \leq p+1$ and we can rewrite $Q^{(p)}$ in the following more condensed form

$$Q^{(p)} = \sum_{1\leq i\leq p+1} Q^{(p,i)}\mathbf{T}_\lambda^{(p,i)} + \pi Q^{(p)} \tag{4.4.19}$$

where

$$\begin{aligned} Q^{(p,i)} &= F^{(p+1,i)}\overline{\partial}\varphi + Q(\partial F^{(\leq p)}, \partial\varphi) + Q(\partial F^{(\leq p)}, \partial F^{(\leq p)}) \\ &\quad + F^{(\leq p)}\overline{\partial}F^{(\leq p)} + F^{(\leq p)}\overline{\partial}\psi^{(\leq p)} + Q(\partial F^{(\leq p)}, \partial\psi^{(\leq p)}) \end{aligned}$$

and

$$\pi Q^{(p)} = Q(\partial\psi^{(p)}, \partial\varphi) + Q(\partial\psi^{(\leq p-2)}, \partial\psi^{(\leq p-2)}) + Q(\partial F^{(\leq p)}, \partial F^{(\leq p)}) + F^{(\leq p)}\overline{\partial}F^{(\leq p)}.$$

Remark 4.4.7. *Here, we used the useful notation $\psi^{(\leq i)}$ to denote any $\psi^{(j)}$ for $j \leq i$. Samewise, $F^{(\leq k)}$ denotes any $F^{(\ell,i)}$ for $1 \leq i \leq \ell \leq k$.*

Recall that our previous remark on the presence of non-oscillating terms basically means that $\pi Q^{(p)} = 0$ if p is odd. Thanks to the decomposition (4.4.19), we see that both sides of (4.4.17) oscillate in a similar manner, and we can assume the following equations on $\psi^{(p)}$ and $F^{(p+1,i)}$:

$$\mathscr{L}F^{(p+1,i)} + \Box F^{(p,i)} = Q^{(p,i)} \quad \text{and} \quad \Box\psi^{(p)} = \pi Q^{(p)}.$$

By collecting the results of this discussion, we define the *reduced system* for the unknowns $(\varphi, \psi^{(\ell)}, F^{(k,i)}, h_\lambda)$ to be

$$\begin{aligned} \Box\varphi &= Q(\partial\varphi, \partial\varphi), && (\mathbf{BG}) \\ \mathscr{L}F^{(k,i)} &= \Box F^{(k-1)} + F^{(k,i)}\overline{\partial}\varphi + Q(\partial F^{(\leq k-1)}, \partial\varphi) + Q(\partial F^{(\leq k-1)}, \partial F^{(\leq k-1)}) && (\mathbf{T}(k,i)) \\ &\quad + F^{(\leq k-1)}\overline{\partial}F^{(\leq k-1)} + F^{(\leq k-1)}\overline{\partial}\psi^{(\leq k-1)} + Q(\partial F^{(\leq k-1)}, \partial\psi^{(\leq k-1)}), \\ \Box\psi^{(k)} &= Q(\partial\psi^{(k)}, \partial\varphi) + Q(\partial\psi^{(\leq k-2)}, \partial\psi^{(\leq k-2)}) + Q(\partial F^{(\leq k)}, \partial F^{(\leq k)}) + F^{(\leq k)}\overline{\partial}F^{(\leq k)}, \\ & && (\mathbf{W}(k)) \\ \Box h_\lambda &= \mathcal{N}(h_\lambda) + \lambda^K Q(\partial F, \partial\varphi)\mathbf{T}_\lambda + \lambda^K\Box F\mathbf{T}_\lambda && (\mathbf{W}) \\ &\quad + \lambda^K\lambda^+\left(F\overline{\partial}F\mathbf{T}_\lambda + Q(\partial F, \partial F)\mathbf{T}_\lambda + Q(\partial F, \partial\psi)\mathbf{T}_\lambda + F\overline{\partial}\psi\mathbf{T}_\lambda + Q(\partial\psi, \partial\psi)\right). \end{aligned}$$

The initial data for $(\varphi, \psi^{(\ell)}, F^{(k,i)}, h_\lambda)$ are given by

$$\begin{aligned} (\varphi, \partial_t\varphi)_{|t=0} &= (\varphi_0, \varphi_1), \\ F^{(k,i)}{}_{|t=0} &= \begin{cases} F_0 & \text{if } (k,i) = (1,1) \\ 0 & \text{if } 2 \leq k \leq K, \end{cases} \\ (\psi^{(k)}, \partial_t\psi^{(k)})_{|t=0} &= (0,0), \\ (h_\lambda, \partial_t h_\lambda)_{|t=0} &= (0,0). \end{aligned} \tag{4.4.20}$$

We denote by S the system of equation (**BG**)-($\mathbf{T}(k,i)$)-($\mathbf{W}(k)$)-(**W**) where k and i take all the possible values such that $F^{(k,i)}$ or $\psi^{(k)}$ are well-defined.

Remark 4.4.8. *Note that in this system, we follow again the convention never to write explicitly the numerical constants appearing in the expansion. But since they appear in Theorem 4.3.1, let us recall the exact equations satisfied by φ and $F^{(1,1)}$:*

$$\Box\varphi_i = Q_i(\partial\varphi,\partial\varphi) \quad \text{and} \quad \mathscr{L}F_i^{(1,1)} = \frac{1}{2}\sum_{k,\ell=1,\dots,d} F_k^{(1,1)} Q_i(\partial(t-r),\partial\varphi_\ell),$$

where Q_i are the null forms of the main system for Φ_λ.

Remark 4.4.9. *As we explained in the discussion before the definition of the reduced system, the functions $\psi^{(k)}$ only exist for even k but for the sake of clarity, it seems useful to forget this and consider that we want to solve the equations (**W**(k)) for all values of k between 2 and $K-1$.*

We see that there is a coupling between (**T**(k,i)) and (**W**(k)) but with a particular triangular structure: the RHS of (**T**(k,i)) involves only $F^{(\ell+1,j)}$ and $\psi^{(\ell)}$ for $\ell \leq k-1$, and the RHS of (**W**(k)) involves only $F^{(\ell,j)}$ and $\psi^{(\ell)}$ for $\ell \leq k$. This is a consequence of our choice of ansatz (4.4.1) and it allows us to solve this system with the following strategy:

1. We first solve (**BG**), this is done in Section 4.5.1.
2. We solve (**T**(1,1)), which initiates a strong induction argument:
   (a) if we know $F^{(\ell,j)}$ and $\psi^{(\ell)}$ for $1 \leq j \leq \ell \leq k$, we can solve (**T**$(k+1,i)$) for $1 \leq i \leq k+1$...
   (b) ... and then solve (**W**$(k+1)$).

   This is done in Section 4.5.2.
3. Finally in Section 4.5.3 we solve (**W**).

Following this strategy, we prove the following theorem about the reduced system S:

Theorem 4.4.1. *Let $K \geq 4$, $N \geq 6+5K$, $R > 0$, $-\frac{1}{2} < \delta < \frac{1}{2}$, $0 < \alpha < 1$, $(\varphi_0,\varphi_1) \in H_\delta^{N+1} \times H_{\delta+1}^N$ and $F_0 \in C^{N-4}$ with support in A_0^R such that*

$$\|F_0\|_{C^{N-4}} + \|\varphi_0\|_{H_\delta^{N+1}} + \|\varphi_1\|_{H_{\delta+1}^N} \leq \varepsilon. \tag{4.4.21}$$

There exists $\varepsilon_0 = \varepsilon_0(K,N,R,\delta) > 0$ such that if $\varepsilon \leq \varepsilon_0$ and $0 < \lambda \leq 1$ there exists a global solution $(\varphi,\psi^{(\ell)},F^{(k,i)},h_\lambda)$ to the reduced system S with initial data as in (4.4.20). Moreover, the following estimates hold:

- *$\varphi \in H^{N+1}$ satisfies*

  $$\left\| w^{\frac{1}{2}}\partial Z^I\varphi \right\|_{L^2} \lesssim \varepsilon \qquad \text{for} \quad |I| \leq N,$$

  with

  $$w(q) := \begin{cases} 1+(1+|q|)^{-\alpha} & \text{for} \quad q<0 \\ 1+(1+|q|)^{2(\delta+1)} & \text{for} \quad q>0 \end{cases},$$

- *$F^{(k,i)} \in C^{N+4-5K}$ is supported in A^R and satisfies*

  $$|Z^I F^{(k,i)}| \lesssim \frac{\varepsilon}{(1+r)}, \qquad \text{for} \quad |I| \leq N+4-5K,$$

- $\psi^{(\ell)} \in H^{N+9-5K}$ *is supported in* B^R *and satisfies*

$$\left\| \partial Z^I \psi^{(\ell)} \right\|_{L^2} \lesssim \varepsilon, \qquad \text{for} \quad |I| \leq N+8-5K,$$

- $h_\lambda \in H^{N+3-5K}$ *is supported in* B^R *and satisfies*

$$\left\| \partial Z^I h_\lambda \right\|_{L^2} \lesssim \varepsilon \lambda^{K-|I|}, \qquad \text{for} \quad |I| \leq N+2-5K.$$

Note that in this article, $\|f\|_{L^2}$ denotes the L^2 norm on a $t = constant$ slice. The rest of this article is devoted to the proof of this theorem, but we can already see that it implies our main result, i.e Theorem 4.3.1.

Proof of Theorem 4.3.1. The reduced system is defined such that if an ansatz of the form (4.4.1) solves it, then Φ_λ solves $\Box \Phi_\lambda = Q(\partial\Phi_\lambda, \partial\Phi_\lambda)$. Therefore, Theorem 4.4.1 implies Theorem 4.3.1, setting

$$\begin{aligned}
\tilde{F}_{0,\lambda} &:= \frac{1}{\lambda} \sum_{\substack{2\leq k\leq K\\ 1\leq i\leq k}} \lambda^k \partial_t F^{(k,i)}{}_{|t=0} \mathbf{T}_\lambda^{(k,i)}{}_{|t=0},\\
F &:= F^{(1,1)},\\
\tilde{F}_\lambda &:= \frac{1}{\lambda^2}\left(\sum_{2\leq k\leq K-1} \lambda^k \psi^{(k)} + \sum_{\substack{2\leq k\leq K\\ 1\leq i\leq k}} \lambda^k F^{(k,i)} \mathbf{T}_\lambda^{(k,i)} + h_\lambda \right).
\end{aligned}$$

As mentioned in Theorem 4.3.1, we want $\tilde{F}_{0,\lambda}$ to depend only on $(\varphi_0, \varphi_1, F_0)$. This holds since the quantities $\partial_t F^{(k,i)}{}_{|t=0}$ can be recover through ($\mathbf{T}(k,i)$) since $\partial_t = \mathscr{L} - \partial_r - \frac{1}{r}$. Because of the triangular structure of the reduced system, it can be seen quite easily that $\partial_t F^{(k,i)}{}_{|t=0}$ only depends on $(\varphi_0, \varphi_1, F_0)$ and their spatial derivatives. □

Remark 4.4.10. *According to Klainerman's result, considering quadratic non-linearities with a null structure ensures global existence for the non-linear wave equation in space dimension 3. But from a high-frequency perspective, null forms are also of prime interest. Indeed, if* $Q(\partial u, \partial v)$ *were not a null form, the expansion of* $Q(\partial(f\mathbf{T}), \partial(g\mathbf{T}))$ *would contain a term of the form* $fgQ(\partial(t-r), \partial(t-r))$*. Looking at the decomposition of* $Q(\partial\Phi_\lambda, \partial\Phi_\lambda)$*, we see that the equation on* $\Phi^{(1)}$ *would be of the form*

$$\mathscr{L}\partial_\theta \Phi^{(1)} = \left(\partial_\theta \Phi^{(1)}\right)^2 + \cdots \tag{4.4.22}$$

where ∂_θ *corresponds to derivatives with respect to the third variable of* $\Phi^{(1)}$ *(see (4.1.4) for the formal definition of* $\Phi^{(1)}$*). The non-linear equation (4.4.22) requires* $\Phi^{(1)}$ *to be described by a "full" Fourier series, i.e*

$$\Phi^{(1)}(t,x,\theta) = \sum_{\ell\in\mathbb{Z}} F^{(1,\ell)}(t,x) \exp(i\ell\theta)$$

with an infinite amount of $F^{(1,\ell)}$ *non-zero. Though the equations for the higher order terms (i.e* $\Phi^{(i)}$ *for* $i \geq 1$*) would remain linear equations, the triangular structure and the presence of* $\Phi^{(1)}$ *as source term would also imply that all the* $\Phi^{(i)}$ *are described by full Fourier series. Therefore the null structure allows us to consider a "simple" ansatz, the so-called half-chessboard ansatz.*

Remark 4.4.11. *The weight w is parametrized by its exponents in the exterior region $q > 0$ and in the interior region $q < 0$. The exterior exponent $2(\delta + 1)$ is chosen so that the bound $\left\| w^{\frac{1}{2}} \partial Z^I \varphi \right\|_{L^2} \lesssim \varepsilon$ holds initially and follows from the smallness assumptions (4.4.21). The constraint on δ then implies that $1 < 2(\delta + 1) < 3$, which will allow us to apply Proposition 4.2.2. The interior exponent α is chosen so that there exists $0 < \nu \leq 1$ and $2 - \nu > \alpha + 1$ (take for example $\nu = \frac{1-\alpha}{2}$), the latter implying*

$$\frac{w}{(1 + |q|)^{2-\nu}} \lesssim w'. \tag{4.4.23}$$

4.5 Global existence for the reduced system

In this section, we prove Theorem 4.4.1 following the strategy outlined previously.

4.5.1 The background wave equation

We first study the global existence for (**BG**). From the celebrated work of Klainermann on null forms, we know that a global solution exists. However, since the resolution of (**W**) will be a sort of high-frequency equivalent of this one, it seems helpful to the author to write down the complete argument leading to the global existence for (**BG**). The proof is a bootstrap argument, based on both L^∞ and L^2 estimates. As mentioned in the introduction, we use Alinhac ghost weight method. Since the equation we solve satisfy the null condition, this is not mandatory. However, the ghost weight method leads to a very efficient proof.

Proposition 4.5.1. *There exists $\varepsilon_0 > 0$ such that, if $\varepsilon \leq \varepsilon_0$, (**BG**) admits a unique global solution φ. Moreover, φ satisfies*

$$\left\| w_1^{\frac{1}{2}} \partial Z^I \varphi \right\|_{L^2} \lesssim \varepsilon \qquad \textit{for} \quad |I| \leq N. \tag{4.5.1}$$

Proof. We define T_0 to be the maximal time of existence of a solution φ, which, by the local in time theory, is positive and satisfies the following blow-up criterion:

$$T_0 < +\infty \iff \lim_{t \to T_0^-} \sum_{|I| \leq N} \left\| w_1^{\frac{1}{2}} \partial Z^I \varphi \right\|_{L^2} (t) = +\infty.$$

Let $C_0 \geq 1$ be a constant to be choosen later. We define $T < T_0$ to be the maximal time such that the inequality

$$\left\| w_1^{\frac{1}{2}} \partial Z^I \varphi \right\|_{L^2} (t) \leq C_0 \varepsilon \qquad\qquad \text{for} \quad |I| \leq N, \tag{4.5.2}$$

hold for $0 \leq t \leq T$. By the assumptions on φ_0 and φ_1 and the Proposition 4.2.1, we have $T > 0$, for C_0 large enough.

We start by deriving decay for $Z^I \varphi$ from (4.5.2). If $|I| \leq N - 3$ we can apply the Klainerman-Sobolev inequality stated in Proposition 4.2.2 (since $1 < 2(\delta + 1) < 3$ and $\alpha > 0$), use the bootstrap assumption (4.5.2) and obtain

$$\left| w_1^{\frac{1}{2}} \partial Z^I \varphi \right| \lesssim \frac{C_0 \varepsilon}{(1 + s)\sqrt{1 + |q|}}.$$

Using the behaviour of the weight w_1 in the exterior and interior region this implies

$$\left|\partial Z^I\varphi\right| \lesssim \begin{cases} \frac{C_0\varepsilon}{(1+s)\sqrt{1+|q|}} & \text{for} \quad q<0 \\ \frac{C_0\varepsilon}{(1+s)(1+|q|)^{\delta+\frac{3}{2}}} & \text{for} \quad q>0 \end{cases}. \tag{4.5.3}$$

In order to derive decay for $Z^I\varphi$, we integrate (4.5.3) along the constant s lines. Recall that the assumptions (4.4.21) and the weighted Sobolev embedding stated in Proposition 4.2.1 implies that

$$\left|\partial Z^I\varphi\right| \lesssim \frac{C_0\varepsilon}{(1+r)^{\delta+\frac{3}{2}}}$$

on the initial hypersurface. Note that $r=s$ if $t=0$. Now, let $q\geq 0$ and integrate (4.5.3) from the initial hypersurface along the constant s lines, this gives

$$\begin{aligned} \left|Z^I\varphi(q\geq 0)\right| &\lesssim \frac{C_0\varepsilon}{(1+s)^{\delta+\frac{3}{2}}} + \int_q^s \frac{C_0\varepsilon \,\mathrm{d}q'}{(1+s)(1+|q'|)^{\delta+\frac{3}{2}}} \\ &\lesssim \frac{C_0\varepsilon}{(1+s)^{\delta+\frac{3}{2}}} + \frac{C_0\varepsilon}{(1+s)(1+|q|)^{\delta+\frac{1}{2}}} \\ &\lesssim \frac{C_0\varepsilon}{(1+s)(1+|q|)^{\delta+\frac{1}{2}}} \end{aligned}$$

where we also used the initial spatial decay and $|q|\leq s$. In particular, we obtain the following decay along the null cone $\left|Z^I\varphi(q=0)\right| \lesssim \frac{C_0\varepsilon}{1+s}$. Now, let $q<0$ and integrate (4.5.3) from the null cone along the constant s lines, this gives

$$\left|Z^I\varphi(q<0)\right| \lesssim \frac{C_0\varepsilon}{1+s} + \int_q^0 \frac{C_0\varepsilon \,\mathrm{d}q'}{(1+s)\sqrt{1+|q'|}} \lesssim C_0\varepsilon\frac{\sqrt{1+|q|}}{1+s}.$$

Since $\delta+\frac{1}{2}>0$, the worst estimate is the interior one and overall we have proved that

$$|Z^I\varphi| \lesssim \frac{C_0\varepsilon\sqrt{1+|q|}}{(1+s)}, \qquad \text{for} \quad |I|\leq N-3. \tag{4.5.4}$$

We then improve the inequality (4.5.2), using the weighted energy estimate given by Lemma 4.2.1. We set

$$\begin{aligned} E(t) &:= \sum_{|I|\leq N} \left\| w_1^{\frac{1}{2}}\partial Z^I\varphi\right\|_{L^2}^2 (t), \\ S(t) &:= \int_0^t \sum_{|I|\leq N} \left\|(w_1')^{\frac{1}{2}}\bar\partial Z^I\varphi\right\|_{L^2}^2 (\tau)\mathrm{d}\tau. \end{aligned}$$

We use Lemma 4.2.1, sum for $|I|\leq N$ and thanks to (4.2.5) and (4.2.6) we obtain for $t\in[0,T]$:

$$E(t)+S(t) \lesssim E(0) + \sum_{|I_1|+|I_2|\leq N} \int_0^t \left\|w_1\partial Z^{I_1}\varphi\bar\partial Z^{I_2}\varphi\partial_t Z^I\varphi\right\|_{L^1}(\tau)\mathrm{d}\tau. \tag{4.5.5}$$

Because of the assumption $(\varphi_0,\varphi_1)\in H^{N+1}_\delta\times H^N_{\delta+1}$ and the choice of exponent in the exterior region for the weight w_1, we have $E(0)\lesssim\varepsilon^2$. Since $|I_1|+|I_2|\leq N$, it must holds that $|I_2|\leq\frac{N}{2}$ or $|I_1|\leq\frac{N}{2}$. We examine the two cases separately:

- If $|I_1| \leq \frac{N}{2}$, then

$$|\partial Z^{I_1}\varphi| \lesssim \frac{1}{1+|q|}|Z^{I_1+1}\varphi| \lesssim \frac{C_0\varepsilon}{(1+s)^{\frac{1}{2}+\frac{\nu}{2}}(1+|q|)^{1-\frac{\nu}{2}}}$$

where we used (4.2.4) and (4.5.4) and where ν is as in the Remark 4.4.11. Using $2ab \leq a^2+b^2$ and (4.5.2) we obtain:

$$\begin{aligned}
&\left\|w_1\partial Z^{I_1}\varphi\bar{\partial}Z^{I_2}\varphi\partial_t Z^I\varphi\right\|_{L^1}(\tau)\\
&\lesssim \frac{C_0\varepsilon}{(1+\tau)^{1+\nu}}\left\|w_1^{\frac{1}{2}}\partial_t Z^I\varphi\right\|_{L^2}^2(\tau) + C_0\varepsilon\left\|\frac{w_1^{\frac{1}{2}}}{(1+|q|)^{1-\frac{\nu}{2}}}\bar{\partial}Z^{I_2}\varphi\right\|_{L^2}^2(\tau)\\
&\lesssim \frac{C_0^3\varepsilon^3}{(1+\tau)^{1+\nu}} + C_0\varepsilon\left\|(w_1')^{\frac{1}{2}}\bar{\partial}Z^{I_2}\varphi\right\|_{L^2}^2(\tau).
\end{aligned}$$

The last step is valid because of (4.4.23).

- If $|I_2| \leq \frac{N}{2}$, then

$$|\bar{\partial}Z^{I_2}\varphi| \lesssim \frac{1}{1+s}|Z^{I_2+1}\varphi| \lesssim \frac{C_0\varepsilon}{(1+s)^{\frac{3}{2}}},$$

where we agin used (4.2.4) and (4.5.4). Using the Cauchy-Schwarz inequality and (4.5.2) we obtain directly:

$$\left\|w_1\partial Z^{I_1}\varphi\bar{\partial}Z^{I_2}\varphi\partial_t Z^I\varphi\right\|_{L^1}(\tau) \lesssim \frac{C_0\varepsilon}{(1+\tau)^{\frac{3}{2}}}\left\|w_1^{\frac{1}{2}}\partial Z^{I_1}\varphi\right\|_{L^2}\left\|w_1^{\frac{1}{2}}\partial_t Z^I\varphi\right\|_{L^2} \lesssim \frac{C_0^3\varepsilon^3}{(1+\tau)^{\frac{3}{2}}}$$

Therefore, since $t \longmapsto (1+\tau)^{-1-\nu}$ and $t \longmapsto (1+\tau)^{-\frac{3}{2}}$ are both integrable over $\mathbb{R}_+$ we obtain for $t \in [0,T]$:

$$\sum_{|I_1|+|I_2|\leq N}\int_0^t \left\|w_1\partial Z^{I_1}\varphi\bar{\partial}Z^{I_2}\varphi\partial_t Z^I\varphi\right\|_{L^1}(\tau)\mathrm{d}\tau \lesssim C_0^3\varepsilon^3 + C_0\varepsilon S(t).$$

Absorbing $C_0\varepsilon S(t)$into the LHS of (4.5.5) we have proved that

$$\left\|w_1^{\frac{1}{2}}\partial Z^I\varphi\right\|_{L^2} \leq C\varepsilon + C\sqrt{C_0\varepsilon}C_0\varepsilon \qquad \text{for} \quad |I| \leq N,$$

with $C > 0$ a numerical constant. We now choose C_0 such that $C \leq \frac{C_0}{4}$ and ε_0 such that $CC_0\varepsilon \leq \frac{C_0}{4}$ and $C\sqrt{C_0\varepsilon} \leq \frac{1}{4}$. Thus, we proved that the inequality (4.5.2) hold with a constant $\frac{C_0}{2}$ instead of C_0, which contradict the maximality of T, thus proving that $T = T_0$. But if $T = T_0$, it implies that the energy is bounded up to the time T_0, implying that $T_0 = +\infty$, which concludes the proof. □

4.5.2 The high-frequency hierarchy

In this section, we solve the coupled transport equations ($\mathbf{T}(k,i)$) and wave equations ($\mathbf{W}(k)$) with a strong induction argument which heavily relies on the triangular structure of the reduced system.

The equations ($\mathbf{T}(k,i)$) are of the form

$$\mathscr{L}f = f\mu + g. \tag{4.5.6}$$

Everything we need to know about (4.5.6) is contained in the following proposition, whose proof is postponed to Appendix 4.B.

Proposition 4.5.2. *Let $M \in \mathbb{N}$, $f_0 : \mathbb{R}^3 \to \mathbb{R}$ a C^M function supported in A_0 such that*

$$\|f_0\|_{C^M} \lesssim \varepsilon.$$

Moreover, let $\mu, g : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}$ two C^M functions with g supported in A^R and such that

$$(1+r)^2|Z^I\mu| + (1+r)^3|Z^I g| \lesssim \varepsilon \qquad \textit{for} \quad |I| \leq M.$$

There exists a unique global solution $f \in C^M$ to (4.5.6) supported in A^R and satisfying

$$|Z^I f| \lesssim \frac{\varepsilon}{1+r} \qquad \textit{for} \quad |I| \leq M.$$

With Lemma (4.5.2) we are now ready to solve ($\mathbf{T}(k,i)$) and ($\mathbf{W}(k)$). Since we will prove that the functions $F^{(k,i)}$ are supported in A^R, we will be only interested in the decay in r, which gives decay in s. We will also forget about q and use many times the fact that in the region A^R, r and t are equivalent. We prove the following proposition:

Proposition 4.5.3. *Let $1 \leq i \leq k \leq K$ and $2 \leq \ell \leq K-1$. There exists $F^{(k,i)} \in C^{N_k}$ supported in A^R solving ($\mathbf{T}(k,i)$) and $\psi^{(\ell)} \in H^{M_\ell}$ supported in B^R solving ($\mathbf{W}(k)$).Moreover, they satisfy the following estimates:*

$$|Z^I F^{(k,i)}| \lesssim \frac{\varepsilon}{1+r}, \qquad \textit{for} \quad |I| \leq N_k, \tag{4.5.7}$$

$$|Z^I \Box F^{(k,i)}| \lesssim \frac{\varepsilon}{(1+r)^3}, \qquad \textit{for} \quad |I| \leq N_k - 2, \tag{4.5.8}$$

$$\left\|\partial Z^I \psi^{(\ell)}\right\|_{L^2} \lesssim \varepsilon, \qquad \textit{for} \quad |I| \leq M_\ell. \tag{4.5.9}$$

where $N_k = N + 4 - 5k$ for $k \geq 2$ and $N_1 = N - 4$ and $M_\ell = N + 3 - 5\ell$.

Remark 4.5.1. *Since $\psi^{(1)}$ is not defined, the limiting term in the equation $\mathbf{T}(2,i)$ is $\Box F^{(1,1)}$, whereas if $k \geq 3$ then the limiting term in the equation $\mathbf{T}(k,i)$ is $\partial\psi^{(k-1)}$ since we lose three derivatives using the Klainerman-Sobolev inequality to obtain decay from L^2 estimates. This explains why $N_1 - N_2 \neq N_k - N_{k+1}$ for all $k \geq 2$.*

To prove Proposition 4.5.3, we proceed by strong induction on the value of k. More precisely, we will show that if the estimates (4.5.7)-(4.5.8)-(4.5.9) holds for $1 \leq k \leq k'$ and $2 \leq \ell \leq k'$ for k' some integer satisfying $2 \leq k' \leq K-2$, then they also hold for $k, \ell = k'+1$. Note that the borderline cases $k = 1$ and $k = K$, for which $\psi^{(k)}$ is not defined, are proved in a similar manner so we don't write down their specific proofs.

Before we start, we derive decay for solutions of wave equation, i.e $\psi^{(\ell)}$ for $\ell \leq k'$ and φ. This follows from the weighted Klainerman-Sobolev inequality of Proposition 4.2.2 and the exact argument has already been given in the proof of Proposition 4.5.1. Following the same strategy, we obtain from (4.5.1) and (4.5.9):

$$|Z^I \psi^{(\ell)}| \lesssim \varepsilon \frac{\sqrt{1+|q|}}{1+s}, \qquad \text{for} \quad |I| \leq M_\ell - 3 \quad \text{and} \quad \ell \leq k', \tag{4.5.10}$$

$$|Z^I \varphi| \lesssim \varepsilon \frac{\sqrt{1+|q|}}{1+s}, \qquad \text{for} \quad |I| \leq N - 3. \tag{4.5.11}$$

4.5.2.1 The transport equation for the oscillating terms

We start by solving $(\mathbf{T}(k'+1,i))$, which we recall:

$$\mathscr{L}F^{(k'+1,i)} = \Box F^{(k')} + F^{(k'+1,i)}\overline{\partial}\varphi + Q(\partial F^{(\leq k')},\partial\varphi) + Q(\partial F^{(\leq k')},\partial F^{(\leq k')}) \\ + F^{(\leq k')}\overline{\partial}F^{(\leq k')} + F^{(\leq k')}\overline{\partial}\psi^{(\leq k')} + Q(\partial F^{(\leq k')},\partial\psi^{(\leq k')}).$$

In sake of clarity we define $G^{(k'+1,i)}$ such that $\mathscr{L}F^{(k'+1,i)} = F^{(k'+1,i)}\overline{\partial}\varphi + G^{(k'+1,i)}$. Note that the expression of $G^{(k'+1,i)}$ involves only $F^{(\ell,j)}$ for $\ell \leq k'$ and $\varphi^{(\ell)}$ for $\ell \leq k'$ so we can estimate it using (4.5.7), (4.5.8) and (4.5.9). This is done in the following lemma.

Lemma 4.5.1. *$G^{(k'+1,i)}$ is supported in A^R and the following estimate holds*

$$|Z^I G^{(k'+1,i)}| \lesssim \frac{\varepsilon}{(1+r)^3}, \qquad \text{for} \quad |I| \leq N-1-5k',$$

Proof. Formally we have

$$G^{(k'+1,i)} = \Box F^{(k')} + Q(\partial F^{(\leq k')},\partial\varphi) + Q(\partial F^{(\leq k')},\partial F^{(\leq k')}) + Q(\partial F^{(\leq k')},\partial\psi^{(\leq k')}).$$

Let us count how many times each terms can be differentiated:

- we want to estimate $\partial\psi^{(\leq k')}$ using (4.5.10), we can differentiate this term at most $M_{k'}-4$ times,
- the terms $\partial F^{(\leq k')}$ are better than $\Box F^{(k')}$ which thanks to (4.5.8) can be differentiated at most $N_{k'}-2$ times.

Since $M_{k'}-4 < N_{k'}-2$, we can differentiate $G^{(k'+1,i)}$ at most $M_{k'}-4$ times, i.e at most $N-1-5k'$ times. Now, let $|I| \leq N-1-5k'$, we have:

$$\begin{aligned}|Z^I G^{(k'+1,i)}| &\lesssim |Z^I \Box F^{(k',i)}| + \sum_{|I_1|+|I_2|\leq |I|+1} \frac{|Z^{I_1}F^{(\leq k')}||Z^{I_2}F^{(\leq k')}|}{1+r} \\ &\quad + \sum_{|I_1|+|I_2|\leq |I|+1} \frac{|Z^{I_1+1}F^{(\leq k')}||Z^{I_2+1}\varphi|}{1+r} + \sum_{|I_1|+|I_2|\leq |I|+1} \frac{|Z^{I_1}F^{(\leq k')}||Z^{I_2}\psi^{(\leq k')}|}{1+r} \\ &\lesssim \frac{\varepsilon}{(1+r)^3},\end{aligned}$$

where we used (4.5.7), (4.5.8), (4.5.10) and (4.5.11). □

Lemma 4.5.2. *There exists a unique solution $F^{(k'+1,i)} \in C^{N-1-5k'}$ to $(\boldsymbol{T}(k'+1,i))$ supported in A^R such that*

$$|Z^I F^{(k'+1,i)}| \lesssim \frac{\varepsilon}{(1+r)}, \qquad \text{for} \quad |I| \leq N-1-5k', \tag{4.5.12}$$

$$|Z^I \Box F^{(k'+1,i)}| \lesssim \frac{\varepsilon}{(1+r)^3}, \qquad \text{for} \quad |I| \leq N-3-5k'. \tag{4.5.13}$$

Proof. We apply Proposition 4.5.2 with $M = N-1-5k'$, $g = G^{(k'+1,i)}$ and $\mu = \overline{\partial}\varphi$ which satisfy the required estimates thanks to Lemma 4.5.1 and to

$$\left|Z^I\overline{\partial}\varphi\right| \lesssim |\overline{\partial}Z^I\varphi| + \sum_{|J|\leq|I|} \frac{|Z^J\varphi|}{1+s} \lesssim \sum_{|J|\leq|I|+1} \frac{|Z^J\varphi|}{1+s} \lesssim \frac{\varepsilon}{(1+r)^2} \tag{4.5.14}$$

in the region A^R for $|I| \leq N-4$ and where we use (4.5.11). This concludes the proof of (4.5.12). Let us now turn to the proof of (4.5.13). Because of Lemma 4.A.2, we have

$$\Box F^{(k'+1,i)} = \delta^{ij}\overline{\partial}_i\overline{\partial}_j F^{(k'+1,i)} + \underline{L}(F^{(k'+1,i)}\overline{\partial}\varphi) + \frac{1}{r}F^{(k'+1,i)}\overline{\partial}\varphi + \underline{L}G^{(k'+1,i)} + \frac{1}{r}G^{(k'+1,i)},$$

where $\underline{L} = \partial_t - \partial_r$. We estimate this expression directly, for I a multi-index such that $|I| \leq N-3-5k'$:

$$\begin{aligned}
|Z^I \Box F^{(k'+1,i)}| &\lesssim |Z^I\overline{\partial}^2 F^{(k'+1,i)}| + |Z^I\partial(F^{(k'+1,i)}\overline{\partial}\varphi)| + |Z^I(r^{-1}F^{(k'+1,i)}\overline{\partial}\varphi)| \\
&\quad + |Z^I\partial G^{(k'+1,i)}| + |Z^I(r^{-1}G^{(k'+1,i)})| \\
&\lesssim \sum_{|J|\leq|I|+2} \frac{|Z^J F^{(k'+1,i)}|}{(1+r)^2} \\
&\quad + \sum_{|I_1|+|I_2|\leq|I|+1} |Z^{I_1}F^{(k'+1,i)}||Z^{I_2}\overline{\partial}\varphi| + \sum_{|J|\leq|I|+1}|Z^J G^{(k'+1,i)}| \\
&\lesssim \frac{\varepsilon}{(1+r)^3}
\end{aligned}$$

where we used commutation property of Z^I and $\overline{\partial}$ as in (4.5.14), the Leibniz rule and $|Z^K(r^{-1})| \lesssim r^{-1}$ for every multi-index K. Using (4.5.12), (4.5.14) and Lemma 4.5.1 we conclude the proof of (4.5.13). $\Box$

Remark 4.5.2. *Note that in the previous proof, we used Lemma 4.A.2, which basically shows that if we have a good control on $\mathscr{L}f$, then $\Box f$ decays better than two derivatives of f, since it behaves like two "good" derivatives of f, i.e $\overline{\partial}^2 f$. This prevents us from estimating $\Box f$ by commuting $\Box$ and $\mathscr{L}$. The decay obtained through this method would not be sufficient since*

$$[\mathscr{L},\Box]f = -\frac{2}{r}\delta^{ij}\partial_i\overline{\partial}_j f + l.o.t$$

where "l.o.t" stands for lower order terms. This would imply that if f solves (4.5.6), then $\Box f$ solves

$$(L-\mu)(r\Box f) = \partial\overline{\partial}f + l.o.t.$$

Since a normal derivative doesn't give extra decay in A^R, we would only get $\Box f \sim \frac{1}{r}$ while we need $\frac{1}{r^3}$, given Proposition 4.5.2.

4.5.2.2 The wave equation for the non-oscillating terms

We now solve $(\mathbf{W}(k'+1))$:

$$\begin{aligned}
\Box\psi^{(k'+1)} &= Q(\partial\psi^{(k'+1)},\partial\varphi) + Q(\partial\psi^{(\leq k'-1)},\partial\psi^{(\leq k'-1)}) \\
&\quad + Q(\partial F^{(\leq k'+1)},\partial F^{(\leq k'+1)}) + F^{(\leq k'+1)}\overline{\partial}F^{(\leq k'+1)}.
\end{aligned}$$

We define $H^{(k'+1)}$ such that $\Box\psi^{(k'+1)} = Q(\partial\psi^{(k'+1)},\partial\varphi) + H^{(k'+1)}$. Note that $H^{(k'+1)}$ involves $\psi^{(\ell)}$ for $\ell \leq k'-1$ and $F^{(\ell,i)}$ for $\ell \leq k'+1$, thus respecting the triangular structure. We estimate $H^{(k'+1)}$ in the following lemma.

Lemma 4.5.3. *$H^{(k'+1)}$ is supported in B^R and the following estimate holds*

$$|Z^I H^{(k'+1)}| \lesssim \frac{\varepsilon^2}{(1+s)^3} \qquad \textit{for} \quad |I| \leq N-2-5k'. \tag{4.5.15}$$

Proof. Formally we have

$$H^{(k'+1)} = Q(\partial\psi^{(\leq k'-1)}, \partial\psi^{(\leq k'-1)}) + Q(\partial F^{(\leq k'+1)}, \partial F^{(\leq k'+1)}) + F^{(\leq k'+1)}\overline{\partial} F^{(\leq k'+1)}.$$

Let us count how many times each terms can be differentiated:

- we want to estimate $\partial\psi^{(\leq k'-1)}$ using (4.5.10), we can differentiate this term at most $M_{k'-1} - 4$ times,

- we want to estimate $\partial F^{(\leq k'+1)}$ using (4.5.12) for $\partial F^{(k'+1)}$ or (4.5.7) for $\partial F^{(\leq k')}$ so we can differentiate these terms at most $N - 2 - 5k'$.

Since $N - 2 - 5k' < M_{k'-1} - 4$, we can differentiate $H^{(k'+1)}$ at most $N - 2 - 5k'$. Now, let $|I| \leq N - 2 - 5k'$, we estimate $Z^I H^{(k'+1)}$ using (4.5.7), (4.5.12) and (4.5.10):

$$\begin{aligned}
|Z^I H^{(2k+2)}| &\lesssim \sum_{|I_1|+|I_2|\leq|I|} \left(\frac{|Z^{I_1+1}F^{(\leq k'+1)}||Z^{I_2+1}F^{(\leq k'+1)}|}{1+r} + \frac{|Z^{I_1+1}\psi^{(\leq k'-1)}||Z^{I_2+1}\psi^{(\leq k'-1)}|}{(1+s)(1+|q|)} \right) \\
&\lesssim \frac{\varepsilon^2}{(1+s)^3}.
\end{aligned}$$

□

Lemma 4.5.4. *If ε is small enough, ($\boldsymbol{W}(k'+1)$) admits a unique global solution $\psi^{(k'+1)} \in H^{N-2-5k'}$ supported in B^R and satisfying*

$$\left\| \partial Z^I \psi^{(k'+1)} \right\|_{L^2} \lesssim \varepsilon \qquad \text{for} \quad |I| \leq N - 2 - 5k'.$$

Proof. In order to prove Lemma 4.5.4 we use a continuity argument. We let T_0 the maximal time of existence of a solution $\psi^{(k'+1)}$ to ($\mathbf{W}(k'+1)$), which satisfies

$$T_0 < +\infty \iff \lim_{t\to T_0^-} \sum_{|I|\leq N-2-5k'} \left\| w^{\frac{1}{2}} \partial Z^I \psi^{(k'+1)} \right\|_{L^2}(t) = +\infty$$

where w is defined in Theorem 4.4.1. Because of the support properties of $H^{(k'+1)}$ and Lemma 4.2.2, the function $\psi^{(k'+1)}$ is supported in B_R (therefore the exterior exponent of w plays no role here). We make the following bootstrap assumption:

$$\left\| w^{\frac{1}{2}} \partial Z^I \psi^{(k'+1)} \right\|_{L^2}(t) \leq \varepsilon \qquad \text{for} \quad |I| \leq N - 2 - 5k'. \tag{4.5.16}$$

Since the initial data for $\psi^{(k'+1)}$ vanish, we can assume that (4.5.16) holds on some time interval $[0, T]$ with $T > 0$. Moreover, we assume that T is the maximal time so that (4.5.16) holds. We define

$$\begin{aligned}
E_{k'+1}(t) &:= \sum_{|I|\leq N-2-5k'} \left\| w^{\frac{1}{2}} \partial Z^I \psi^{(k'+1)} \right\|_{L^2}^2 (t), \\
S_{k'+1}(t) &:= \int_0^t \sum_{|I|\leq N-2-5k'} \left\| (w')^{\frac{1}{2}} \overline{\partial} Z^I \psi^{(k'+1)} \right\|_{L^2}^2 (\tau) \mathrm{d}\tau.
\end{aligned}$$

We apply the weighted energy estimate of Lemma 4.2.1, the Cauchy-Schwarz inequality and our bootstrap assumption (4.5.16) to obtain (recall that the initial data for $\psi^{(k'+1)}$ vanish):

$$\begin{aligned} E_{k'+1}(t) + S_{k'+1}(t) &\lesssim \sum_{|I|\leq N-2-5k'} \int_0^t \left\| w\Box Z^I\psi^{(k'+1)}\partial Z^I\psi^{(k'+1)}\right\|_{L^1(B_R\cap\Sigma_\tau)} \mathrm{d}\tau \\ &\lesssim \sum_{|I|\leq N-2-5k'} \int_0^t \left\| w^{\frac12}\Box Z^I\psi^{(k'+1)}\right\|_{L^2(B_R\cap\Sigma_\tau)} \left\| w^{\frac12}\partial Z^I\psi^{(k'+1)}\right\|_{L^2} \mathrm{d}\tau. \end{aligned} \tag{4.5.17}$$

Now if $|I| \leq N-2-5k'$, using (4.2.5) and (4.2.6) we obtain:

$$\begin{aligned} \left\| w^{\frac12}\Box Z^I\psi^{(k'+1)}\right\|_{L^2(B_R\cap\Sigma_\tau)} &\lesssim \sum_{|J_1|+|J_2|\leq|I|} \left\| w^{\frac12}\partial Z^{J_1}\psi^{(k'+1)}\bar\partial Z^{J_2}\varphi\right\|_{L^2(B^R\cap\Sigma_\tau)} \\ &\quad + \sum_{|J_1|+|J_2|\leq|I|} \left\| w^{\frac12}\bar\partial Z^{J_1}\psi^{(k'+1)}\partial Z^{J_2}\varphi\right\|_{L^2(B^R\cap\Sigma_\tau)} \\ &\quad + \sum_{|J|\leq|I|} \left\| w^{\frac12} Z^J H^{(k'+1)}\right\|_{L^2(B^R\cap\Sigma_\tau)} \\ &=: A+B+C. \end{aligned}$$

We start with the easiest term, i.e C, for which we use (4.5.15) and the fact that w is bounded on B_R:

$$C \lesssim \varepsilon^2 \left(\int_0^{\tau+R} \frac{\mathrm{d}r}{(1+r+\tau)^4}\right)^{\frac12} \lesssim \frac{\varepsilon^2}{(1+\tau)^{\frac32}}$$

which implies $C \lesssim \varepsilon^2$. For A, we use (4.5.11) and $|q| \leq s$ to get

$$|\bar\partial Z^{J_2}\varphi| \lesssim \frac{\varepsilon}{(1+s)^{\frac32}}$$

for all $|J_2| \leq N-2-5k'$.This gives

$$A \lesssim \varepsilon \sum_{|I|\leq N-2-5k'} \frac{1}{(1+\tau)^{\frac32}} \left\| w^{\frac12}\partial Z^I\psi^{(k'+1)}\right\|_{L^2} \lesssim \frac{\varepsilon^2}{(1+\tau)^{\frac32}}$$

where we also used (4.5.16). Let us now look at B, by first using (4.5.11) to obtain

$$|\partial Z^{J_2}\varphi| \lesssim \frac{\varepsilon}{(1+s)^{\frac12+\frac{\nu}{2}}(1+|q|)^{1-\frac{\nu}{2}}}$$

for ν defined at the beginning of this proof. Using (4.4.23) this gives

$$\begin{aligned} B &\lesssim \frac{\varepsilon}{(1+\tau)^{\frac12+\frac{\nu}{2}}} \sum_{|J|\leq|I|} \left\| \frac{w^{\frac12}}{(1+|q|)^{1-\frac{\nu}{2}}}\bar\partial Z^J\psi^{(k'+1)}\right\|_{L^2(B^R\cap\Sigma_\tau)} \\ &\lesssim \frac{\varepsilon}{(1+\tau)^{\frac12+\frac{\nu}{2}}} \sum_{|J|\leq|I|} \left\| (w')^{\frac12}\bar\partial Z^J\psi^{(k'+1)}\right\|_{L^2(B^R\cap\Sigma_\tau)}. \end{aligned}$$

Putting our estimates on A, B and C together we obtain from (4.5.17):

$$\begin{aligned}
&E_{k'+1}(t) + S_{k'+1}(t) \\
&\lesssim \sum_{|I|\leq N-2-5k'} \int_0^t (A+C) \left\| w^{\frac{1}{2}} \partial Z^I \psi^{(k'+1)} \right\|_{L^2} \mathrm{d}\tau + \sum_{|I|\leq N-2-5k'} \int_0^t B \left\| w^{\frac{1}{2}} \partial Z^I \psi^{(k'+1)} \right\|_{L^2} \mathrm{d}\tau \\
&\lesssim \varepsilon^3 \int_0^t \frac{\mathrm{d}\tau}{(1+\tau)^{\frac{3}{2}}} \\
&\quad + \sum_{|J|\leq |I|\leq N-2-5k'} \int_0^t \frac{\varepsilon}{(1+\tau)^{\frac{1}{2}+\frac{\nu}{2}}} \left\| (w')^{\frac{1}{2}} \bar{\partial} Z^J \psi^{(k'+1)} \right\|_{L^2(B^R \cap \Sigma_\tau)} \left\| w^{\frac{1}{2}} \partial Z^I \psi^{(k'+1)} \right\|_{L^2} \mathrm{d}\tau \\
&\lesssim \varepsilon^3 \int_0^t \frac{\mathrm{d}\tau}{(1+\tau)^{\frac{3}{2}}} \\
&\quad + \sum_{|I|\leq N-2-5k'} \left(\varepsilon \int_0^t \left\| (w')^{\frac{1}{2}} \bar{\partial} Z^I \psi^{(k'+1)} \right\|^2_{L^2(B^R\cap\Sigma_\tau)} \mathrm{d}\tau + \varepsilon \int_0^t \frac{1}{(1+\tau)^{1+\nu}} \left\| w^{\frac{1}{2}} \partial Z^I \psi^{(k'+1)} \right\|^2_{L^2} \mathrm{d}\tau \right) \\
&\lesssim \varepsilon^3 \int_0^t \frac{\mathrm{d}\tau}{(1+\tau)^{\frac{3}{2}}} + \varepsilon S_{k'+1}(t) + \varepsilon^3 \int_0^t \frac{\mathrm{d}\tau}{(1+\tau)^{1+\nu}}
\end{aligned}$$

where for the integral involving $A+C$ we use (4.5.16) and for the integral involving B we used first $2ab \leq a^2 + b^2$ and (4.5.16). Therefore, we have proved that there exists a constant $C > 0$ such that for all $t \in [0, T]$

$$E_{k'+1}(t) + (1 - C\varepsilon) S_{k'+1}(t) \leq C\varepsilon^3.$$

We now choose ε so that $\varepsilon \leq \frac{1}{2C}$. This implies that for all $t \in [0, T]$

$$E_{k'+1}(t) \leq \frac{1}{2}\varepsilon^2.$$

Therefore, we improved (4.5.16), thus proving that actually $T = T_0$. This breaks the blow-up criterion, proving that $T_0 = +\infty$. Note that since $\psi^{(k'+1)}$ is supported in B_R the ghost weight w is bounded and we can change (4.5.16) to the estimate of the lemma. □

This concludes the induction and thus the proof of Proposition 4.5.3.

4.5.3 The equation for the remainder

The proof of the global existence for (**W**) follows the same structure as the proof of Proposition 4.5.1.

4.5.3.1 Bootstrap assumptions and first consequences

We use again the ghost weight w defined in Theorem 4.4.1. We let T_0 be the maximal time of existence of a solution h_λ, which satisfies the following blow-up criterion:

$$T_0 < +\infty \iff \lim_{t\to T_0^-} \sum_{|I|\leq N'} \left\| w^{\frac{1}{2}} \partial Z^I h_\lambda \right\|_{L^2}(t) = +\infty,$$

where N' is some large enough integer. In terms of number of derivation, the limiting term in the equation for h_λ is $\Box F^{(K,i)}$, which we can differentiate at most $N + 2 - 5K$ thanks to

(4.5.8). Therefore we set $N' = N + 2 - 5K$, which satisfies $N' \geq 8$. We define two rescaled energies:

$$\mathcal{E}_\lambda(t) := \sum_{|I| \leq N'} \lambda^{-2K+2|I|} \left\| w^{\frac{1}{2}} \partial Z^I h_\lambda \right\|_{L^2}^2 (t),$$
$$\mathcal{S}_\lambda(t) := \sum_{|I| \leq N'} \lambda^{-2K+2|I|} \int_0^t \left\| (w')^{\frac{1}{2}} \bar{\partial} Z^I h_\lambda \right\|_{L^2}^2 (\tau) \mathrm{d}\tau.$$

We make the following bootstrap assumptions on h_λ:

$$\sup_{t \in [0,T]} \mathcal{E}_\lambda(t) \leq C_2^2 \varepsilon^2. \tag{4.5.18}$$

Here C_2 is a constant to be choosen later. We let $T < T_0$ be the maximal time such that these estimates hold on $[0,T]$. Since $(h_\lambda, \partial_t h_\lambda)_{|t=0} = 0$, this bootstrap assumption is satisfied at $t = 0$ and $T > 0$. Note that (4.5.18) immediately implies that for every multi-index $|I| \leq N'$ and for every $t \in [0,T]$ we have

$$\left\| w^{\frac{1}{2}} \partial Z^I h_\lambda \right\|_{L^2} (t) \leq C_2 \varepsilon \lambda^{K - |I|}. \tag{4.5.19}$$

Recall (**W**):

$$\begin{aligned}\Box h_\lambda &= \mathcal{N}(h_\lambda) + \lambda^K Q(\partial F, \partial \varphi) \mathbf{T}_\lambda + \lambda^K \Box F \mathbf{T}_\lambda \\ &\quad + \lambda^K \lambda^+ \left(F \bar{\partial} F \mathbf{T}_\lambda + Q(\partial F, \partial F) \mathbf{T}_\lambda + Q(\partial F, \partial \psi) \mathbf{T}_\lambda + F \bar{\partial} \psi \mathbf{T}_\lambda + Q(\partial \psi, \partial \psi) \right)\end{aligned}$$

To capture the different support properties, we define

$$\begin{aligned}\mathbf{A}_\lambda &= \lambda^K Q(\partial F, \partial \varphi) \mathbf{T}_\lambda + \lambda^K \Box F \mathbf{T}_\lambda \\ &\quad + \lambda^K \lambda^+ \left(F \bar{\partial} F \mathbf{T}_\lambda + Q(\partial F, \partial F) \mathbf{T}_\lambda + Q(\partial F, \partial \psi) \mathbf{T}_\lambda + F \bar{\partial} \psi \mathbf{T}_\lambda \right) \\ \mathbf{B}_\lambda &= \lambda^K \lambda^+ Q(\partial \psi, \partial \psi)\end{aligned}$$

such that

$$\Box h_\lambda = \mathcal{N}(h_\lambda) + \mathbf{A}_\lambda + \mathbf{B}_\lambda. \tag{4.5.20}$$

Thanks to the vanishing of the initial data of h_λ and the fact that the RHS of (4.5.20) is supported in B_R, we can apply Lemma 4.2.2 to prove that h_λ is also supported in B_R.

4.5.3.2 Decay estimates

We start by deriving some decay for h_λ and the non-homogeneous terms in $\Box h_\lambda$. This again follows from the weighted Klainerman-Sobolev inequality and we don't repeat the argument. However, we state it in a lemma to emphasize the appearance of the factor $\lambda^{K-3-|I|}$.

Lemma 4.5.5. *The following estimate holds on* $[0,T]$

$$|Z^I h_\lambda| \lesssim \frac{C_2 \varepsilon \sqrt{1 + |q|}}{1 + s} \lambda^{K-3-|I|} \qquad \textit{for} \quad |I| \leq N' - 3. \tag{4.5.21}$$

The following lemma estimates the non-homogeneous terms in $\Box h_\lambda$.

Lemma 4.5.6. *The function* $\mathbf{A}_\lambda$ *are supported in* A^R *and satisfies*

$$|Z^I \mathbf{A}_\lambda| \lesssim \frac{\varepsilon\lambda^{K-|I|}}{(1+s)^3}, \qquad \textit{for} \quad |I| \leq N'. \tag{4.5.22}$$

The function $\mathbf{B}_\lambda$ *is supported in* B^R *and satisfies*

$$|Z^I \mathbf{B}_\lambda| \lesssim \frac{\varepsilon^2}{(1+s)^3}\lambda^{K+1} \qquad \textit{for} \quad |I| \leq N'. \tag{4.5.23}$$

Proof. Since

$$|Z^I \mathbb{T}_\lambda^{(\alpha,\beta)}| \leq C_{I,\alpha,\beta}\lambda^{-|I|}, \tag{4.5.24}$$

for some $C_{I,\alpha,\beta} > 0$, we can directly estimate $\mathbf{A}_\lambda$ using its expression:

$$\begin{aligned}
|Z^I \mathbf{A}_\lambda| \lesssim & \sum_{|I_1|+|I_2|\leq |I|} \lambda^{K-|I_2|}|Z^{I_1}\Box F| \\
& + \sum_{|I_1|+|I_2|+|I_3|\leq |I|} \lambda^{K-|I_3|}\frac{|Z^{I_1+1}F|}{1+r}\left(|Z^{I_2+1}F| + |Z^{I_2+1}\psi| + |Z^{I_2+1}\varphi|\right) \\
\lesssim & \frac{\varepsilon}{(1+s)^3}\sum_{|J|\leq |I|}\lambda^{K-|J|}
\end{aligned}$$

where we used (4.5.7), (4.5.8),(4.5.10) and (4.5.24). For $\mathbf{B}_\lambda$ we recall (4.5.10), which implies that for all $2 \leq \ell \leq K-1$

$$|Z^I \psi^{(\ell)}| \lesssim \varepsilon\frac{\sqrt{1+|q|}}{1+s}, \qquad \text{for} \quad |I| \leq M_{K-1} - 3.$$

Recalling the expression of $\mathbf{B}_\lambda$ concludes the proof:

$$|Z^I \mathbf{B}_\lambda| \lesssim \lambda^{K+1}\sum_{|I_1|+|I_2|\leq |I|}\frac{|Z^{I_1+1}\psi||Z^{I_2+1}\psi|}{(1+s)(1+|q|)} \lesssim \frac{\varepsilon^2}{(1+s)^3}\lambda^{K+1}$$

since $N' + 1 \leq M_{K-1} - 3$. □

4.5.3.3 Energy estimates

We conclude the bootstrap by improving (4.5.18).

Proposition 4.5.4. *For large enough* C_2 *and small enough* ε *(depending on* C_2*), we have for every* $t \in [0, T]$

$$\mathcal{E}_\lambda(t) \leq \frac{1}{2}C_2^2\varepsilon^2.$$

Proof. Applying Lemma 4.2.1 to $Z^I h_\lambda$ for every multi-index $|I| \leq N'$, multiplying what we obtain by $\lambda^{-2K+2|I|}$ and summing over I we get for every $t \in [0, T]$ (recall that the initial data for h_λ and $\partial_t h_\lambda$ vanish):

$$\begin{aligned}
\mathcal{E}_\lambda(t) + \mathcal{S}_\lambda(t) \lesssim & \sum_{|I|\leq N'}\int_0^t \lambda^{-2K+2|I|}\left\|w\Box Z^I h_\lambda \partial Z^I h_\lambda\right\|_{L^1}\mathrm{d}\tau \\
\lesssim & \sum_{|I|\leq N'}\int_0^t \lambda^{-2K+2|I|}\left\|w^{\frac{1}{2}}\Box Z^I h_\lambda\right\|_{L^2}\left\|w^{\frac{1}{2}}\partial Z^I h_\lambda\right\|_{L^2}\mathrm{d}\tau
\end{aligned} \tag{4.5.25}$$

where we used the Cauchy-Schwarz inequality. Thanks to (4.5.20) and (4.2.5) we get for $|I| \leq N'$:

$$\left\|w^{\frac{1}{2}}\Box Z^I h_\lambda\right\|_{L^2} \lesssim \sum_{|J|\leq|I|}\left(\left\|w^{\frac{1}{2}}Z^J\mathcal{N}(h_\lambda)\right\|_{L^2(B_R\cap\Sigma_\tau)} + \left\|Z^J\mathbf{A}_\lambda\right\|_{L^2(A_R\cap\Sigma_\tau)} + \left\|Z^J\mathbf{B}_\lambda\right\|_{L^2(B_R\cap\Sigma_\tau)}\right).$$

Thanks to Lemma 4.5.6 we have

$$\left\|Z^J\mathbf{A}_\lambda\right\|_{L^2(A_R\cap\Sigma_\tau)} + \left\|Z^J\mathbf{B}_\lambda\right\|_{L^2(B_R\cap\Sigma_\tau)} \lesssim \varepsilon\lambda^{K-|J|}\left(\int_0^{\tau+R}\frac{\mathrm{d}r}{(1+r+\tau)^4}\right)^{\frac{1}{2}} \lesssim \frac{\varepsilon\lambda^{K-|J|}}{(1+\tau)^{\frac{3}{2}}}. \tag{4.5.26}$$

We now look at $\mathcal{N}(h_\lambda)$ using (4.4.16):

$$\left\|w^{\frac{1}{2}}Z^J\mathcal{N}(h_\lambda)\right\|_{L^2} \lesssim \sum_{\substack{|J_1|+|J_2|\leq|J| \\ \Upsilon\in\{\varphi,\psi,\lambda^+F\mathbf{T}_\lambda\} \\ j=1,2}}\left(N_1^{J_1,J_2} + N_{\Upsilon,j}^{J_1,J_2}\right)$$

with

$$\begin{aligned} N_1^{J_1,J_2} &= \left\|w^{\frac{1}{2}}\partial Z^{J_1}h_\lambda\overline{\partial}Z^{J_2}h_\lambda\right\|_{L^2}, \\ N_{\Upsilon,1}^{J_1,J_2} &= \left\|w^{\frac{1}{2}}\partial Z^{J_1}h_\lambda\overline{\partial}Z^{J_2}\Upsilon\right\|_{L^2}, \\ N_{\Upsilon,2}^{J_1,J_2} &= \left\|w^{\frac{1}{2}}\partial Z^{J_1}\Upsilon\overline{\partial}Z^{J_2}h_\lambda\right\|_{L^2}. \end{aligned}$$

We start by $N_1^{J_1,J_2}$ and we distinguish two cases: since $|J_1|+|J_2| \leq N'$ we have either $|J_1| \leq \frac{N'}{2}$ or $|J_2| \leq \frac{N'}{2}$.

- If $|J_1| \leq \frac{N'}{2}$, then (4.5.21) implies (since $N' \geq 8 \Longrightarrow \frac{N'}{2}+1 \leq N'-3$)

$$|\partial Z^{J_1}h_\lambda| \lesssim \frac{\varepsilon\lambda^{K-4-|J_1|}}{(1+s)^{\frac{1}{2}+\frac{\nu}{2}}(1+|q|)^{1-\frac{\nu}{2}}}$$

  where ν is as in (4.4.23). Using (4.4.23) we obtain

$$N_1^{J_1,J_2} \lesssim \frac{\varepsilon\lambda^{K-4-|J_1|}}{(1+\tau)^{\frac{1}{2}+\frac{\nu}{2}}}\left\|(w')^{\frac{1}{2}}\overline{\partial}Z^{J_2}h_\lambda\right\|_{L^2}. \tag{4.5.27}$$

- If $|J_2| \leq \frac{N'}{2}$, then (4.5.21) implies

$$\overline{\partial}Z^{J_2}h_\lambda \lesssim \frac{\varepsilon\lambda^{K-4-|J_2|}}{(1+s)^{\frac{3}{2}}}.$$

  Therefore using (4.5.19) we obtain

$$N_1^{J_1,J_2} \lesssim \frac{\varepsilon\lambda^{K-4-|J_2|}}{(1+\tau)^{\frac{3}{2}}}\left\|w^{\frac{1}{2}}\partial Z^{J_1}h_\lambda\right\|_{L^2} \lesssim \frac{\varepsilon^2\lambda^{2K-4-|J_1|-|J_2|}}{(1+\tau)^{\frac{3}{2}}}. \tag{4.5.28}$$

We now turn to $N^{J_1,J_2}_{\Upsilon,1}$ for $\Upsilon \in \{\varphi,\psi,\lambda^+ F\mathbf{T}_\lambda\}$. Using (4.5.10), (4.5.11) and (4.5.7) we obtain

$$|\bar\partial Z^{J_2}\Upsilon| \lesssim \frac{\varepsilon\lambda^{-|J_2|}}{(1+s)^{\frac{3}{2}}}$$

where we used the presence of λ^+ to regain one power of λ. Using then (4.5.19) this implies

$$N^{J_1,J_2}_{\Upsilon,1} \lesssim \frac{\varepsilon\lambda^{-|J_2|}}{(1+\tau)^{\frac{3}{2}}}\left\|w^{\frac12}\partial Z^{J_1}h_\lambda\right\|_{L^2} \lesssim \frac{\varepsilon^2\lambda^{K-|J_1|-|J_2|}}{(1+\tau)^{\frac{3}{2}}}. \tag{4.5.29}$$

Finally, let us look at $N^{J_1,J_2}_{\Upsilon,2}$ for $\Upsilon \in \{\varphi,\psi,\lambda^+ F\mathbf{T}_\lambda\}$. Using again (4.5.10), (4.5.11) and (4.5.7) we obtain

$$|\partial Z^{J_1}\Upsilon| \lesssim \frac{\varepsilon\lambda^{-|J_1|}}{(1+s)^{\frac12+\frac{\nu}{2}}(1+|q|)^{1-\frac{\nu}{2}}}$$

where ν is as in (4.4.23). Therefore using (4.4.23) we obtain

$$N^{J_1,J_2}_{\Upsilon,2} \lesssim \frac{\varepsilon\lambda^{-|J_1|}}{(1+\tau)^{\frac12+\frac{\nu}{2}}}\left\|(w')^{\frac12}\bar\partial Z^{J_2}h_\lambda\right\|_{L^2}. \tag{4.5.30}$$

We now put (4.5.26)-(4.5.27)-(4.5.28)-(4.5.29)-(4.5.30) together into (4.5.25) to obtain

$$\begin{aligned}
\mathcal{E}_\lambda(t)+\mathcal{S}_\lambda(t) \lesssim &\sum_{|J|\leq|I|\leq N'}\int_0^t \frac{\varepsilon\lambda^{-K+2|I|-|J|}}{(1+\tau)^{\frac32}}\left\|w^{\frac12}\partial Z^I h_\lambda\right\|_{L^2}\mathrm{d}\tau\\
&+\sum_{|J_1|+|J_2|\leq|I|\leq N'}\int_0^t \frac{\varepsilon^2\lambda^{-K+2|I|-|J_1|-|J_2|}}{(1+\tau)^{\frac32}}\left\|w^{\frac12}\partial Z^I h_\lambda\right\|_{L^2}\mathrm{d}\tau\\
&+\sum_{|J_1|+|J_2|\leq|I|\leq N'}\int_0^t \frac{\varepsilon\lambda^{-2K+2|I|-|J_1|}}{(1+\tau)^{\frac12+\frac{\nu}{2}}}\left\|(w')^{\frac12}\bar\partial Z^{J_2}h_\lambda\right\|_{L^2}\left\|w^{\frac12}\partial Z^I h_\lambda\right\|_{L^2}\mathrm{d}\tau
\end{aligned}$$

where the first line corresponds to (4.5.26), the second to (4.5.28) and (4.5.29) and the third to (4.5.27) and (4.5.30). Note that we used $K\geq 4$ to bound λ^{K-4} by 1 in (4.5.27) and (4.5.28). For the first two integrals, we simply use (4.5.19). For the third integral we use $2ab\leq a^2+b^2$ and split the λ powers carefully:

$$\begin{aligned}
&\int_0^t \frac{\varepsilon\lambda^{-2K+2|I|-|J_1|}}{(1+\tau)^{\frac12+\frac{\nu}{2}}}\left\|(w')^{\frac12}\bar\partial Z^{J_2}h_\lambda\right\|_{L^2}\left\|w^{\frac12}\partial Z^I h_\lambda\right\|_{L^2}\\
&\quad\lesssim \int_0^t \varepsilon\lambda^{-2K+2|J_2|}\left\|(w')^{\frac12}\bar\partial Z^{J_2}h_\lambda\right\|^2_{L^2} + \int_0^t \frac{\varepsilon\lambda^{-2K+4|I|-2|J_1|-2|J_2|}}{(1+\tau)^{1+\nu}}\left\|w^{\frac12}\partial Z^I h_\lambda\right\|^2_{L^2}\\
&\quad\lesssim \varepsilon\mathcal{S}_\lambda(t)+\varepsilon^3\lambda^{2(|I|-|J_1|-|J_2|)}
\end{aligned}$$

Since $|J_1|+|J_2|\leq|I|$ and $|J|\leq|I|$ we have proved that there exists a constant $C>0$ such that

$$\mathcal{E}_\lambda(t)+(1-C\varepsilon)\mathcal{S}_\lambda(t)\leq CC_2\varepsilon^2+C\varepsilon^3$$

We now choose $C_2\geq 4C$ and $\varepsilon\leq\frac{C_2^2}{4C}$ so that for every $t\in[0,T]$, $\mathcal{E}_\lambda(t)\leq\frac{C_2^2}{2}\varepsilon^2$.

□

The result of Proposition 4.5.4 contradicts the maximality of T and thus $T=T_0$. But now $\mathcal{E}_\lambda(t)$ is bounded up to T_0 by $C_2^2\varepsilon^2$, which proves that $T_0=+\infty$. This concludes the proof of Theorem 4.4.1.

Appendix

4.A Commutators

In this first appendix, we estimate $[Z,\overline{\partial}]$ and $[Z,\mathscr{L}]$. The later will be used in the second appendix, where we solve equation (4.5.6).

Lemma 4.A.1. *For every $Z\in\mathcal{Z}$, we have*

$$[Z,\overline{\partial}_i]=p_i^{\ell}\overline{\partial}_\ell+\sum_{Z'\in\mathcal{Z}}\frac{p_{i,Z'}}{r}Z',$$

for some functions p_i^ℓ and $p_{i,Z'}$ such that for every $Z'\in\mathcal{Z}$, multi-index K and $\ell\in\{1,2,3\}$ there exists $C_Z^{K,\ell}$ such that

$$|Z^K p_i^\ell|+|Z^K p_{i,Z'}|\leq C_J^{K,I,\ell}$$

in the region A^R.

Proof. It follows from direct computations, using the formula $\overline{\partial}_i=\partial_i-\frac{x_i}{r}\partial_r$:

$$\begin{aligned}
[\overline{\partial}_i,\partial_t]&=0,\\
[\overline{\partial}_i,\partial_j]&=\frac{\omega_i}{r}\overline{\partial}_j+\frac{1}{r}(\delta_{ij}-\omega_i\omega_j)\partial_r,\\
[\overline{\partial}_i,S]&=\overline{\partial}_i,\\
[\overline{\partial}_i,\Omega_{jk}]&=\delta_{ik}\overline{\partial}_j-\delta_{ij}\overline{\partial}_k,\\
[\overline{\partial}_i,\Omega_{0j}]&=\frac{t\omega_i}{r}\overline{\partial}_j+\frac{\omega_k}{r}(\delta_{ij}-\omega_i\omega_j)\Omega_{0k}.
\end{aligned}$$

□

The following proposition estimates the commutator between the Minkowski vector fields and the transport operator $\mathscr{L}$.

Proposition 4.A.1. *For every multi-index I we have*

$$[\mathscr{L},Z^I]=\sum_{|J|\leq|I|-1}\left(\frac{a_J^{I,\ell}}{r}\overline{\partial}_\ell Z^J+b_J^IZ^J\mathscr{L}+\frac{c_J^I}{r^2}Z^J\right),$$

for some functions $a_J^{I,\ell}$, b_J^I and c_J^I such that for every multi-index I, J and K with $|J|+1\leq|I|$ and every $\ell\in\{1,2,3\}$ there exists $C_J^{K,I,\ell}>0$ such that

$$|Z^Ka_J^{I,\ell}|+|Z^Kb_J^I|+|Z^Kc_J^I|\leq C_J^{K,I,\ell}$$

in the region A^R.

Proof. We prove this by strong induction on the value of $|I|$. Some direct computations show that

$$\begin{aligned}[\mathscr{L},\partial_t]=[\mathscr{L},\Omega_{ij}]&=0,\\ [\mathscr{L},\partial_i]&=-\frac{1}{r}\overline{\partial}_i+\frac{\omega_i}{r^2}\mathrm{Id},\\ [\mathscr{L},\Omega_{0i}]&=\frac{t-r}{r}\overline{\partial}_i-\omega_i\mathscr{L}+\frac{\omega_i(r-t)}{r^2}\mathrm{Id},\\ [\mathscr{L},S]&=\mathscr{L},\end{aligned}$$

which proves the proposition if $|I|=1$. We use the notation $[\mathscr{L},Z]=\frac{a_0^{Z,\ell}}{r}\overline{\partial}_\ell+b_0^Z\mathscr{L}+\frac{c_0^Z}{r^2}\mathrm{Id}$. Now let $|I|$ be a multi-index with $|I|\leq N-1$ and Z be any minkowskian vector field, we have

$$\begin{aligned}[\mathscr{L},ZZ^I]&=Z[\mathscr{L},Z^I]+[\mathscr{L},Z]Z^I\\ &=\sum_{|J|\leq|I|-1}\left(Z\left(\frac{a_J^{I,\ell}}{r}\right)\overline{\partial}_\ell Z^J+Z\left(b_J^I\right)Z^J\mathscr{L}+Z\left(\frac{c_J^I}{r^2}\right)Z^J\right)\\ &\quad+\sum_{|J|\leq|I|}\left(b_J^IZ^J\mathscr{L}+\frac{c_J^I}{r^2}Z^J\right)\\ &\quad+\sum_{|J|\leq|I|-1}\frac{a_J^{I,\ell}}{r}Z\overline{\partial}_\ell Z^J+\frac{a_0^{Z,\ell}}{r}\overline{\partial}_\ell Z^I+b_0^Z\mathscr{L}Z^I+\frac{c_0^Z}{r^2}Z^I.\end{aligned}$$

Using the assumptions made on the functions $a_J^{I,\ell}$, b_J^I and c_J^I, the fact that $|Z^K(r^{-1})|\lesssim r^{-1}$ for every multi-index K, we see that the first sum contains only good terms, meaning that they have the form we want. It is also the case for the terms in the second sum and for the terms $\frac{a_0^{Z,\ell}}{r}\overline{\partial}_\ell Z^I$ and $\frac{c_0^Z}{r^2}Z^I$. To conclude the proof, it remains to deal with the third sum and $b_0^Z\mathscr{L}Z^I$:

- for $b_0^Z\mathscr{L}Z^I$, we simply write $b_0^Z\mathscr{L}Z^I=b_0^ZZ^I\mathscr{L}+b_0^Z[\mathscr{L},Z^I]$. Because of the assumptions made on b_0^Z, it proves that the term $b_0^Z\mathscr{L}Z^I$ has the desired form.
- for the third sum, we use Lemma 4.A.1:

$$\begin{aligned}\sum_{|J|\leq|I|-1}\frac{a_J^{I,\ell}}{r}Z\overline{\partial}_\ell Z^J&=\sum_{|J|\leq|I|}\frac{a_J^{I,\ell}}{r}\overline{\partial}_\ell Z^J+\sum_{|J|\leq|I|-1}\frac{a_J^{I,\ell}}{r}[Z,\overline{\partial}_\ell]Z^J\\ &=\sum_{|J|\leq|I|}\frac{a_J^{I,\ell}}{r}\overline{\partial}_\ell Z^J+\sum_{|J|\leq|I|-1}\frac{a_J^{I,\ell}p_\ell^k}{r}\overline{\partial}_kZ^J+\sum_{|J|\leq|I|-1}\frac{a_J^{I,\ell}p_{\ell,Z'}}{r^2}Z'Z^J,\end{aligned}$$

  which is of the desired form.

This conclude the proof of the proposition. □

The following Lemma estimates $\Box f$ if we know $\mathscr{L}f$.

Lemma 4.A.2. *If f and g are such that $\mathscr{L}f=g$, then*

$$\Box f=\delta^{ij}\overline{\partial}_i\overline{\partial}_jf-\underline{L}g-\frac{g}{r}.$$

Proof. Applying separately $-\partial_t + \partial_r$ to $\mathscr{L} f = g$ we obtain

$$-\partial_t^2 f + \partial_r^2 f - \frac{1}{r}\left(\partial_t f - \partial_r f + \frac{f}{r}\right) = -\underline{L} g.$$

We recognize on the LHS of the previous equation the operator $\mathscr{L}$ applied to f so that :

$$-\partial_t^2 f + \partial_r^2 f + \frac{2}{r}\partial_r f = -\underline{L} g - \frac{g}{r}.$$

Recalling that $\Delta = \partial_r^2 + \frac{2}{r}\partial_r + \delta^{ij}\overline{\partial}_i\overline{\partial}_j$ concludes the proof. □

4.B The transport equations

The goal of this appendix is to prove Proposition 4.5.2.

4.B.1 The reduced transport equation

We start by considering the following reduced transport equation

$$(L + \eta) f = g \tag{4.B.1}$$

where

- $g : \mathbb{R}_+ \times \mathbb{R}^3 \longmapsto \mathbb{R}$ is a continuous function supported in A^R,
- $\eta : \mathbb{R}_+ \times \mathbb{R}^3 \longmapsto \mathbb{R}$ is a continuous function such that $|\eta| \lesssim \frac{\varepsilon}{(1+s)^2}$.

We start by constructing a well-chosen solution to $(L - \eta)\phi = 0$ which is close to 1.

Definition 4.B.1. *Let $\chi : \mathbb{R}^3 \to [0,1]$ be a smooth function supported in $\{\frac{1}{2R} \leq r \leq 2R\}$ and such that $\chi_{|A_0} \equiv 1$. Using polar coordinates we define $\hat{\chi}$ a cut-off function around A^R in $\mathbb{R}_+ \times \mathbb{R}^3$ by*

$$\hat{\chi}(t, r, \omega) = \chi(r - t, \omega).$$

Lemma 4.B.1. *There exists a function β vanishing outside of $\{\frac{1}{2R} \leq q \leq 2R\}$ which satisfies*

$$|\beta| \lesssim \varepsilon \qquad \text{and} \qquad L(1+\beta) = (1+\beta)\eta \quad \text{in } A^R.$$

Proof. We simply define β using polar coordinates

$$\beta(t, r, \omega) := \exp\left(\hat{\chi}(t, r, \omega)\int_0^t \eta(t', r - t + t', \omega)\mathrm{d}t'\right) - 1.$$

Thanks to our assumption on η, β satisfies the bound $|\beta| \lesssim \varepsilon$. Thus, if ε is small enough, we can bound $1 + \beta$ from below by $\frac{1}{2}$, and therefore $\ln(1+\beta)$ is well-defined and satisfies clearly $L(\ln(1+\beta)) = \eta$ in A^R, which implies $L(1+\beta) = (1+\beta)\eta$. □

It is now easy to solve (4.B.1):

Lemma 4.B.2. *Let $f_0 : \mathbb{R}^3 \to \mathbb{R}$ supported in A_0 and $g : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}$ supported in A^R. There exists a unique global solution f of (4.B.1) with initial data f_0. This solution is supported in A^R and satisfies*

$$|f| \lesssim \sup_{(t,x)\in\mathbb{R}_+\times\mathbb{R}^3} \left(|f_0(x)| + (1+|x|)^2|g(t,x)|\right).$$

Proof. The uniqueness follows easily from an energy estimate related to the operator $L+\eta$. To construct a solution, we use polar coordinates and set

$$f(t,r,\omega):=\frac{1}{1+\beta(t,r,\omega)}\Bigg((1+\beta(0,r-t,\omega))f_0(r-t,\omega)\\ +\int_0^t(1+\beta(s,r-t+s,\omega))g(s,r-t+s,\omega)\mathrm{d}s\Bigg).$$

By construction, we have $f_{|t=0}=f_0$ and $L((1+\beta)f)=(1+\beta)g$, and f is supported in A^R. Let $(t,x)\in A^R$, since $L(1+\beta)=(1+\beta)\eta$ in A^R we have:

$$(L+\eta)f(t,x)=\frac{1}{1+\beta(t,x)}L((1+\beta)f)(t,x)=g(t,x).$$

This proves the existence of a solution. The estimate on $|f|$ comes from the expression of f, the estimate $\frac{1}{2}\le 1+\beta\le 2$ and the support property of g (note that in A^R, r and t are equivalent). □

4.B.2 The transport operator $\mathscr{L}$

Using the result of the previous section, we now solve the equation

$$\mathscr{L}f=f\mu+g \tag{4.B.2}$$

where

$$(1+r)^2|Z^I\mu|+(1+r)^3|Z^Ig|\lesssim\varepsilon \tag{4.B.3}$$

for $|I|\le M$, where M is an integer. We consider initial data f_0 supported in A_0 and satisfying $\|f_0\|_{C^M}\lesssim\varepsilon$. Our strategy is to first deduce some *a priori* estimates.

We first take advantage of $[\mathscr{L},\Omega]=0$:

Lemma 4.B.3. *If f is a smooth and global in time solution of (4.B.2) initially supported in A_0, then*

$$|\Omega^Jf|\lesssim\frac{\varepsilon}{1+r}\qquad \text{for}\quad |J|\le M. \tag{4.B.4}$$

Proof. We prove this lemma by strong induction on the value of $|J|$. If $|J|=0$, we just rewrite (4.B.2) as

$$(L-\mu)(rf)=rg$$

and apply Lemma 4.B.2 using (4.B.3). Assume now that the estimate (4.B.4) holds for all value of $|J|\le k$ with $k\le M-1$ and let J be a multi-index such that $|J|=k+1$. Since $[\mathscr{L},\Omega]=0$, the equation for Ω^Jf is

$$\mathscr{L}\Omega^Jf=\Omega^Jf\mu+\sum_{\substack{J_1+J_2=J\\|J_1|\le k}}\Omega^{J_1}f\Omega^{J_2}\mu+\Omega^Jg$$

which rewrites

$$(L-\mu)(r\Omega^Jf)=\sum_{\substack{J_1+J_2=J\\|J_1|\le k}}r\Omega^{J_1}f\Omega^{J_2}\mu+r\Omega^Jg.$$

Because of the induction hypothesis and of (4.B.3), the RHS of this equation is bounded by $\frac{\varepsilon}{(1+r)^2}$. Thus, we can apply Lemma 4.B.2 and conclude the proof. □

Note that in the previous lemma, in order to apply Lemma 4.B.2, we have to prove that the initial data for $r\Omega^J f$ are bounded. Since the vector fields Ω involve only spatial derivations, this boundedness property follows from f being supported in A_0 and smooth. In the next lemma however, we fully use Proposition 4.A.1 and estimate $\Omega^J Z^I f$, again with the help of Lemma 4.B.2. This requires to bound the initial data for $r\Omega^J Z^I f$, which may involve some time derivative. To find $\partial_t^k f_{|t=0}$, we simply use the equation (4.B.2) and $[\mathscr{L}, \partial_t] = 0$ to derive the equation for $\partial_t^{k-1} f$ which gives

$$\partial_t^k f = -\partial \partial_t^{k-1} f - \frac{\partial_t^{k-1} f}{r} + \sum_{i+j=k-1} \partial_t^i f \partial_t^j \mu + \partial_t^{k-1} g.$$

Therefore we obtain $\partial_t^k f_{|t=0}$ by induction on k, and check easily their initial boundedness. With that in mind, we can estimate $\Omega^J Z^I f$:

Lemma 4.B.4. *If f is a smooth and global in time solution of (4.B.2) initially supported in A_0, then*

$$|\Omega^J Z^I f| \lesssim \frac{\varepsilon}{1+r} \qquad \textit{for} \quad |I| + |J| \leq M. \tag{4.B.5}$$

Proof. We prove this lemma by strong induction on the value of $|I|$. The case $|I| = 0$ is a consequence of Lemma 4.B.3. Assume now that the estimate (4.B.5) holds for all value of $|I'| \leq k$ with $k \leq M-1$ and all J' such that $|I'| + |J'| \leq M$. Let I be a multi-index such that $|I| = k+1$ and let J be a multi-index such that $|I| + |J| \leq M$. Formally, the equation on $\Omega^J Z^I f$ is

$$(L-\mu)(r\Omega^J Z^I f) = \sum_{\substack{I_1+I_2=I \\ J_1+J_2=J \\ |I_1|\leq k \\ |J_1| \leq |J|-1}} r\Omega^{J_1} Z^{I_1} f \Omega^{J_2} Z^{I_2} \mu + r\Omega^J Z^I g + r[\mathscr{L}, \Omega^J Z^I] f. \tag{4.B.6}$$

Using the induction hypothesis and (4.B.3), the first two terms in the RHS of (4.B.6) are bounded by $\frac{\varepsilon}{(1+r)^2}$. It remains to estimate the commutator. Using $[\mathscr{L}, \Omega] = 0$ and Proposition 4.A.1 we have:

$$\begin{aligned} |r[\mathscr{L}, \Omega^J Z^I] f| &\lesssim \left| r\Omega^J \sum_{|I'| \leq |I|-1} \left(\frac{a_{I'}^{I,\ell}}{r} \overline{\partial}_\ell Z^{I'} f + b_{I'}^I Z^{I'} \mathscr{L} f + \frac{c_{I'}^I}{r^2} Z^{I'} f \right) \right| \\ &\lesssim \sum_{|I'| \leq |I|-1} \left| \Omega^J \overline{\partial} Z^{I'} f \right| + \sum_{|I'| \leq |I|-1} r \left| \Omega^J Z^{I'} \mathscr{L} f \right| + \sum_{|I'| \leq |I|-1} \frac{1}{r} \left| \Omega^J Z^{I'} f \right| \end{aligned} \tag{4.B.7}$$

Since $|I'| \leq |I| - 1$ we can use the induction hypothesis to obtain

$$\sum_{|I'| \leq |I|-1} \frac{1}{r} \left| \Omega^J Z^{I'} f \right| \lesssim \frac{\varepsilon}{(1+r)^2}.$$

To estimate the terms involving $\mathscr{L} f$, we simply use (4.B.2), the induction hypothesis and (4.B.3):

$$\sum_{|I'| \leq |I|-1} r \left| \Omega^J Z^{I'} \mathscr{L} f \right| \lesssim \sum_{\substack{|I'| \leq |I|-1 \\ I_1+I_2=I' \\ J_1+J_2=J}} r|\Omega^{J_1} Z^{I_1} f \Omega^{J_2} Z^{I_2} \mu| + r|\Omega^J Z^I g| \lesssim \frac{\varepsilon}{(1+r)^2}.$$

It remains to estimate the first terms in (4.B.7). Using $[\Omega, \overline{\partial}] \sim \overline{\partial}$ and $\overline{\partial}_\ell = \frac{\omega^k}{r}\Omega_{\ell k}$ we get

$$\sum_{|I'| \leq |I|-1} \left|\Omega^J \overline{\partial} Z^{I'} f\right| \lesssim \sum_{\substack{|I'| \leq |I|-1 \\ |J'| \leq |J|+1}} \frac{1}{r} \left|\Omega^{J'} Z^{I'} f\right|$$

In this last sum, note that $|I'| \leq |I| - 1$ and $|I'| + |J'| \leq |I| - 1 + |J| + 1 \leq M$. Thus we can use our induction hypothesis to bound this by $\frac{\varepsilon}{(1+r)^2}$. In conclusion, the RHS of (4.B.6) is bounded by $\frac{\varepsilon}{(1+r)^2}$, and we can apply Lemma 4.B.2 and conclude the proof. □

In the previous lemma, the case $J = 0$ gives that a global smooth solution of (4.B.2) satisfies

$$|Z^I f| \lesssim \frac{\varepsilon}{1+r} \qquad \text{for} \quad |I| \leq M.$$

This *a priori* estimates allows us easily to prove the existence and uniqueness of a global solution to (4.B.2). We summarize this discussion in the following corollary:

Corollary 4.B.1. *Let f_0 supported in A_0 and such that and let μ and g satisfy (4.B.3). There exists a unique global solution $f \in C^M$ to (4.B.2) with initial data f_0. Moreover, f is supported in A^R and satisfies*

$$|Z^I f| \lesssim \frac{\varepsilon}{1+r} \qquad \textit{for} \quad |I| \leq M.$$

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ECOLE
DOCTORALE
DE MATHEMATIQUES
HADAMARD

Titre : Construction d'espace-temps haute-fréquences

Mots clés : relativité générale, équations d'Einstein, équations aux dérivées partielles, optique géométrique faiblement non-linéaire, problème de Cauchy, équations de contraintes, ondes gravitationnelles

Résumé : Cette thèse s'intéresse à des solutions haute-fréquences aux équations de la relativité générale. Ces solutions permettent de décrire la propagation d'ondes gravitationnelles dans un contexte non-linéaire. Elles sont aussi un exemple de backreaction et contribuent ainsi à l'étude de la conjecture de Burnett.
Les Chapitres 2 et 3 sont dédiés à la construction de solutions locales en temps haute-fréquences $(g_\lambda)_\lambda$ aux équations d'Einstein dans le vide en jauge d'onde généralisée, approchant en un sens faible une métrique background g_0 modélisant une poussière nulle. Les données initiales pour les équations d'Einstein devant vérifier les équations de contraintes, on construit des solutions haute-fréquences à ces équations sur $\mathbb{R}^3$ dans le Chapitre 2 en adaptant la méthode conforme, i.e en définissant ses paramètres par des ansatz haute-fréquences.
Dans le Chapitre 3, on construit la famille $(g_\lambda)_\lambda$, qui est définie par un ansatz haute-fréquence. Pour la partie oscillante de l'ansatz, on identifie une hiérarchie d'équations de transport linéaires ainsi que des conditions de polarisations. Ces dernières sont propagées par les identités de Bianchi, et elles permettent de contrôler la création d'harmoniques grâce à la structure nulle polarisée faible des termes semi-linéaires. Le couplage avec l'équation d'onde quasi-linéaire pour le reste induit une perte de dérivée que l'on résout en utilisant le double feuilletage nul de la métrique background et en introduisant un cut-off en Fourier. On évite ainsi de construire le double feuilletage nul pour les métriques g_λ.
Le Chapitre 4 est dédié à l'étude en temps long d'une famille de solutions haute-fréquences à une équation d'onde semi-linéaire présentant une structure nulle. En se basant sur la méthode des champs de vecteurs, on montre l'existence globale de ces solutions.

Title : Construction of high-frequency spacetimes

Keywords : general relativity, Einstein equations, partial differential equations, weakly non-linear geometric optics, Cauchy problem, constraint equations, gravitational waves

Abstract : This thesis is interested in high-frequency solutions to the equations of general relativity. These solutions describe the propagation of gravitational waves in a non-linear context. They also display the backreaction phenomenon and thus contribute to the study of Burnett's conjecture.
Chapters 2 and 3 are devoted to the construction of high-frequency local in time solutions $(g_\lambda)_\lambda$ to the Einstein vacuum equations in generalised wave coordinates, approaching in a weak sense a background null-dust solution. The initial data for the Einstein vacuum must satisfy the constraint equations and we construct high-frequency solutions of these equations on $\mathbb{R}^3$ in Chapter 2 by adapting the conformal method, where the parameters are defined by high-frequency ansatz.
In Chapter 3, we construct the family $(g_\lambda)_\lambda$, defined by a high-frequency ansatz. For the oscillating part of the ansatz, we identify a hierarchy of transport equations as well as polarization conditions. The later are propagated through the Bianchi identities, and allow us to control the creation of harmonics thanks to the weak polarized null structure of the semi-linear terms. The coupling with the quasi-linear wave equation for the remainder induces a loss of derivatives which we regain using the background double null foliation and a Fourier cut-off. Thus, we avoid constructing the double null foliation for the metrics g_λ.
Chapter 4 is devoted to the long time study of high-frequency solutions to a semi-linear wave equation with a null structure. We show global existence using the vector field method.

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