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ABSTRACT

In this paper, a direct rigorous mathematical proof of the Gregory–Laflamme instability for the five-dimensional Schwarzschild black string is presented. Under a choice of ansatz for the perturbation and a gauge choice, the linearized vacuum Einstein equation reduces to an ordinary differential equation (ODE) problem for a single function. In this work, a suitable rescaling and change of variables is applied, which casts the ODE into a Schrödinger eigenvalue equation to which an energy functional is assigned. It is then shown by direct variational methods that the lowest eigenfunction gives rise to an exponentially growing mode solution, which has admissible behavior at the future event horizon and spacelike infinity. After the addition of a pure gauge solution, this gives rise to a regular exponentially growing mode solution of the linearized vacuum Einstein equation in harmonic/transverse-traceless gauge.

I. INTRODUCTION

The main topic of this paper is the study of the stability problem for the Schwarzschild black string solution to the Einstein vacuum equation in five dimensions. In 1993, the work of Gregory–Laflamme[1] gave strong numerical evidence for the presence of an exponentially growing mode instability. This phenomenon has since been known as the Gregory–Laflamme instability. This work has been widely invoked in the physics community to infer instability of many higher dimensional spacetimes, for example, black rings, ultraspinning Myers–Perry black holes, and black Saturns. The interested reader should consult Refs. 2 and 3 and references therein, as well as Ref. 4 and Refs. 5 and 6, which give a general approach to stability problems. The purpose of the present paper is to provide a direct, self-contained, and elementary mathematical proof of the Gregory–Laflamme instability of the $5D$ Schwarzschild black string.

A. Schwarzschild black holes, black strings, and black branes

The most basic solution to the vacuum Einstein equation

$$\mathrm{Ric}_g = 0 \tag{1.1}$$

giving rise to the black hole phenomena is the Schwarzschild black hole solution (Sch_n, g_s). It arises dynamically as the maximal Cauchy development of the following initial data: an initial hypersurface $\Sigma_0 = \mathbb{R} \times \mathbb{S}^{n-2}$, a first fundamental form (in isotropic coordinates)

$$h_s = \left(1 + \frac{M}{2\rho^{n-3}}\right)^{\frac{4}{n-3}} (d\rho \otimes d\rho + \rho^2 \mathring{\gamma}_{n-2}), \qquad \rho \in (0, \infty) \cong \mathbb{R}, \tag{1.2}$$

and the second fundamental form $K = 0$, where $\mathring{\gamma}_{n-2}$ is the metric on the unit $(n-2)$-sphere $\mathbb{S}^{n-2}$. This spacetime is asymptotically flat and spherically symmetric. The Penrose diagram in Fig. 1 represents the causal structure of (Sch_n, g_s) arising from this initial data, restricted to the

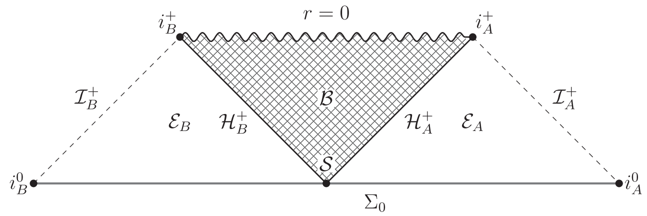

FIG. 1. The Penrose diagram of the Schwarzschild spacetime (Sch_n, g_s). Here, $\mathcal{I}^+ := \mathcal{I}_A^+ \cup \mathcal{I}_B^+$ is future null infinity, $i^+ := i_A^+ \cup i_B^+$ and $i^0 := i_A^0 \cup i_B^0$ are future timelike infinity and spacelike infinity, respectively, $\mathcal{E}_A := J^-(\mathcal{I}_A^+) \cap J^+(\Sigma_0)$ is the distinguished exterior region, $\mathcal{E}_B := J^-(\mathcal{I}_B^+) \cap J^+(\Sigma_0)$ is another exterior region, $\mathcal{B} := \mathrm{Sch}_n \backslash J^-(\mathcal{I}^+)$ is the black hole region, $\mathcal{H}^+ = \mathcal{H}_A^+ \cup \mathcal{H}_B^+ := \mathcal{B}\backslash \mathrm{int}(\mathcal{B})$ is the future event horizon, and $\mathcal{S} := \mathcal{H}_A^+ \cap \mathcal{H}_B^+$ is the bifurcation sphere. The wavy line denotes a singular boundary, which is not part of the spacetime (Sch_n, g_s) but towards which the Kretchmann curvature invariant diverges. It is in this sense that (Sch_n, g_s) is singular. Note that every point in this diagram is, in fact, an $(n-2)$-sphere.

future of Σ_0. The metric on the exterior $\mathcal{E}_A$ (see Fig. 1) of the n-dimensional Schwarzschild black hole in traditional Schwarzschild coordinates $(t, r, \varphi_1, \ldots, \varphi_{n-2})$ takes the form

$$g_s = -D_n(r) dt \otimes dt + \frac{1}{D_n(r)} dr \otimes dr + r^2 \mathring{\gamma}_{n-2}, \qquad D_n(r) = 1 - \frac{2M}{r^{n-3}}, \tag{1.3}$$

where $t \in [0, \infty)$, $r \in \left((2M)^{\frac{1}{n-3}}, \infty\right)$, and $\mathring{\gamma}_{n-2}$ is the metric on the unit $(n-2)$-sphere.

The Lorentzian manifold that is the main topic of this paper is the Schwarzschild black string spacetime in five dimensions, which is constructed from the $4D$ Schwarzschild solution (Sch_4, g_s). Before focusing on this spacetime explicitly, it is of interest to discuss more general spacetimes constructed from the n-dimensional Schwarzschild black hole solution (Sch_n, g_s). Let $\mathbb{S}_R^1$ denote the circle of radius R, and let $\mathrm{F}_p \in \{\mathbb{R}^p, \mathbb{R}^{p-1} \times \mathbb{S}_R^1, \ldots, \mathbb{R} \times \prod_{i=1}^{p-1} \mathbb{S}_{R_i}^1, \prod_{i=1}^{p} \mathbb{S}_{R_i}^1\}$ with its associated p-dimensional Euclidean metric δ_p. If one has the n-dimensional Schwarzschild black hole spacetime (Sch_n, g_s) and takes its Cartesian product with F_p, then one realizes the $(n+p)$-dimensional Schwarzschild black brane $(\mathrm{Sch}_n \times \mathrm{F}_p, g_s \oplus \delta_p)$. This means that the $(n+p)$-dimensional Schwarzschild black brane $(\mathrm{Sch}_n \times \mathrm{F}_p, g_s \oplus \delta_p)$ is a product manifold made from Ricci-flat manifolds, which is again Ricci-flat and hence satisfies the vacuum Einstein equation (1.1). Note that in contrast to (Sch_n, g_s), the spacetimes $(\mathrm{Sch}_n \times \mathrm{F}_p, g_s \oplus \delta_p)$ are not asymptotically flat but are called "asymptotically Kaluza–Klein."

The Schwarzschild black brane spacetimes $(\mathrm{Sch}_n \times \mathrm{F}_p, g_s \oplus \delta_p)$ arise dynamically as the maximal Cauchy development of suitably extended Schwarzschild initial data, i.e., $(\Sigma_0 \times \mathrm{F}_p, h_s \oplus \delta_p, K = 0)$. Hence, the above Penrose diagram in Fig. 1 can be reinterpreted as the Penrose diagram for the Schwarzschild black brane, but instead of each point representing a $(n-2)$-sphere, it represents a $\mathbb{S}^{n-2} \times \mathrm{F}_p$. In particular, the notation $\mathcal{E}_A$ will be used henceforth to denote the distinguished exterior region of $(\mathrm{Sch}_n \times \mathrm{F}_p, g_s \oplus \delta_p)$.

Taking $p = 1$ gives rise to the $(n+1)$-dimensional Schwarzschild black string spacetime $\mathrm{Sch}_n \times \mathbb{R}$ or alternatively $\mathrm{Sch}_n \times \mathbb{S}_R^1$. The topic of the present paper is the $5D$ Schwarzschild black string spacetime $\mathrm{Sch}_4 \times \mathbb{R}$ or alternatively $\mathrm{Sch}_4 \times \mathbb{S}_R^1$. The metric on the exterior $\mathcal{E}_A$ in standard Schwarzschild coordinates is

$$g := -D(r) dt \otimes dt + \frac{1}{D(r)} dr \otimes dr + r^2 \mathring{\gamma}_2 + dz \otimes dz, \qquad D(r) = 1 - \frac{2M}{r}, \tag{1.4}$$

where $t \in [0, \infty)$, $r \in (2M, \infty)$, and $z \in \mathbb{R}$ or $\mathbb{R}/2\pi R\mathbb{Z}$.

Finally, to analyze the subsequent problem of linear stability on the exterior region $\mathcal{E}_A$ up to the future event horizon $\mathcal{H}_A^+$, one requires a chart with coordinate functions that are regular up to this hypersurface $\mathcal{H}_A^+ \backslash \mathcal{S}$, where $\mathcal{S}$ now denotes the bifurcation surface. A good choice is ingoing Eddington–Finkelstein coordinates defined by

$$v = t + r_*, \qquad \frac{dr_*}{dr} = \frac{r^{n-3}}{r^{n-3} - 2M}, \qquad \text{with } r_*(3M) = 3M + 2M \log(M). \tag{1.5}$$

The $(n+p)$-dimensional Schwarzschild black brane metric becomes

$$g_s \oplus \delta = -D_n(r) dv \otimes dv + dv \otimes dr + dr \otimes dv + r^2 \mathring{\gamma}_{n-2} + \delta_{ij} dz^i \otimes dz^j, \qquad D_n(r) = 1 - \frac{2M}{r^{n-3}}. \tag{1.6}$$

B. Previous works

For a good introduction to the Gregory–Laflamme instability and the numerical result of Ref. 1, see Ref. 7. A detailed survey of the key work[8] related to the present paper is undertaken in Sec. III. A brief history of the problem is presented here:

1. In 1988, Gregory–Laflamme examined the Schwarzschild black string spacetime and stated that it is stable.[9] However, an issue in the analysis arose from working in Schwarzschild coordinates, which lead to incorrect regularity assumptions for the asymptotic solutions.
2. In 1993, Gregory–Laflamme used numerics to give strong evidence for the existence of a low-frequency instability of the Schwarzschild black string and branes in harmonic gauge.[1]
3. In 1994, Gregory–Laflamme generalized their numerical analysis to show instability of "magnetically-charged dilatonic" black branes[10] (see Refs. 10 and 11 for a discussion of these solutions).
4. In 2000, Gubser–Mitra discussed the Gregory–Laflamme instability for general black branes. They conjectured that a necessary and sufficient condition for stability of the black brane spacetimes is thermodynamic stability of the corresponding black hole.[12,13]
5. In 2000, Reall,[14] with the aim of addressing the Gubser–Mitra conjecture, explored further the relation between the stability of black branes arising from static, spherically symmetric black holes and thermodynamic stability of those black holes. In particular, the work of Reall argues that there is a direct relation between the "negative mode" of the Euclidean Schwarzschild instanton solution (this mode was initially identified in a paper by Gross, Perry, and Yaffe[15]) and the threshold of the Gregory–Laflamme instability. This idea was further explored in a work of Reall *et al.*,[16] which extended the idea that "negative modes" of the Euclidean extension of a Myers–Perry black hole (the generalization of the Kerr spacetime to higher dimensions, see Refs. 17 and 2 for details) correspond to the threshold for the onset of a Gregory–Laflamme instability.
6. In 2006, Hovdebo and Myers[8] used a different gauge (which was introduced in Ref. 18) to reproduce the numerics from the original work of Gregory and Laflamme. This gauge choice will be called spherical gauge and will be adopted in the present work. This work discusses the presence of the Gregory–Laflamme instability for the "boosted" Schwarzschild black string and the Emparan–Reall black ring (for a discussion of this solution, see Refs. 2, 19, and 20).
7. In 2010, Lehner and Pretorius numerically simulated the non-linear evolution of the Gregory–Laflamme instability; see the review[21] and references therein.
8. In 2011, Figueras, Murata, and Reall[4] put forward the idea that a local Penrose inequality gives a stability criterion. Furthermore, Ref. 4 showed numerically that this local Penrose inequality was violated for the Schwarzschild black string for a range of frequency parameters, which closely match those found in the original work of Gregory–Laflamme.[1]
9. In 2012, Hollands and Wald[5] and, later in 2015, Prabu and Wald[6] developed a general method applicable to many linear stability problems, which encompasses the problem of linear stability of the Schwarzschild black string exterior $\mathcal{E}_A$. References 5 and 6 are explored in detail in Sec. I E.

C. Statement of the main theorem

The purpose of this paper is to give a direct, self-contained, elementary proof of the Gregory–Laflamme instability for the $5D$ Schwarzschild black string.

For the statement of the main theorem, one should have in mind the Penrose diagram in Fig. 2 for the $5D$ Schwarzschild black string spacetime.

Definition 1.1 (Mode Solution). A solution of the linearized vacuum Einstein equation

$$g^{cd}\nabla_c\nabla_d h_{ab} + \nabla_a\nabla_b h - 2\nabla_{(b}\nabla^c h_{a)c} + 2R_a{}^c{}_b{}^d h_{cd} = 0 \tag{1.7}$$

on the exterior $\mathcal{E}_A$ of the Schwarzschild black string $\mathrm{Sch}_4\times\mathbb{R}$ *of the form*

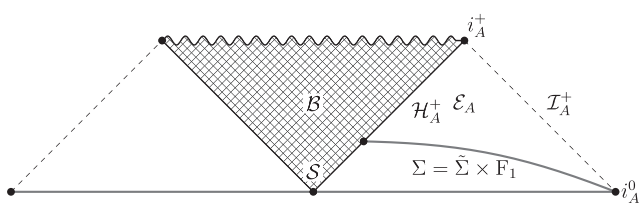

FIG. 2. The Penrose diagram for the $5D$ Schwarzschild black string illustrating the set up for the linear instability problem. Indicated is a spacelike asymptotically flat hypersurface $\tilde{\Sigma}$, which extends from spacelike infinity i_A^0 to intersect the future event horizon $\mathcal{H}_A^+$ to the future of the bifurcation surface $\mathcal{S}$. Furthermore, $\mathrm{F}_1 = \mathbb{R}$ or $\mathbb{S}^1_R$, $\mathcal{B}$ is the black hole region, $\mathcal{E}_A$ is the exterior region, $\mathcal{I}_A^+$ is future null infinity, and i_A^+ is future timelike infinity. The hypersurface Σ can be expressed as $\Sigma = \{(t, r_*, \theta, \varphi, z) : t = f(r_*)\}$ such that $f\sim 1$ for $r_*\to\infty$. An explicit example would be a hypersurface of constant t_*, where $t_* = t + 2M\log(r-2M)$.

$$h_{\alpha\beta} = e^{\mu t + ikz} H_{\alpha\beta}(r,\theta) \tag{1.8}$$

with $\mu, k \in \mathbb{R}$ and $(t, r, \theta, \varphi, z)$ *standard Schwarzschild coordinates will be called a mode solution of (1.7).*

A way of establishing the linear instability of an asymptotically flat black hole is exhibiting a mode solution of the linearized Einstein equation (1.7), which is smooth up to and including the future event horizon and decays toward spacelike infinity and such that $\mu > 0$.

Theorem 1.1 (Gregory–Laflamme Instability). *For all* $|k| \in [\frac{3}{20M}, \frac{8}{20M}]$, *there exists a non-trivial mode solution h of the form (1.8) to the linearized vacuum Einstein equation (1.7) on the exterior* $\mathcal{E}_A$ *of the Schwarzschild black string background* $\mathrm{Sch}_4 \times \mathbb{R}$ *with* $\mu > \frac{1}{40\sqrt{10M}} > 0$ *and*

$$H_{\alpha\beta}(r,\theta) = \begin{pmatrix} H_{tt}(r) & H_{tr}(r) & 0 & 0 & 0 \\ H_{tr}(r) & H_{rr}(r) & 0 & 0 & 0 \\ 0 & 0 & H_{\theta\theta}(r) & 0 & 0 \\ 0 & 0 & 0 & H_{\theta\theta}(r)\sin^2\theta & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1.9}$$

The solution h extends regularly to $\mathcal{H}_A^+$ *and decays exponentially towards* i_A^0 *and can thus be viewed as arising from regular initial data on a hypersurface* Σ *extending from the future event horizon* $\mathcal{H}_A^+$ to i_A^0. *In particular,* $h|_\Sigma$ *and* $\nabla h|_\Sigma$ *are smooth on* Σ. *Moreover, the solution h is not pure gauge and can, in fact, be chosen such that the harmonic/transverse-traceless gauge conditions*

$$\begin{cases} \nabla^a h_{ab} = 0, \\ g^{ab} h_{ab} = 0 \end{cases} \tag{1.10}$$

are satisfied.

Suppose $R > 4M$, *then one can choose k such that there exists an integer* $n \in [\frac{3R}{20M}, \frac{8R}{20M}]$, *and therefore, h induces a smooth solution on the exterior* $\mathcal{E}_A$ *of the Schwarzschild black string* $\mathrm{Sch}_4 \times \mathbb{S}^1_R$. *Moreover, the initial data for such a mode solution on the exterior* $\mathcal{E}_A$ *of* $\mathrm{Sch}_4 \times \mathbb{S}^1_R$ *have finite energy.*

Hence, the exterior $\mathcal{E}_A$ *of the Schwarzschild black string* $\mathrm{Sch}_4 \times \mathbb{R}$ *or* $\mathrm{Sch}_4 \times \mathbb{S}^1_R$ *for* $R > 4M$ *is linearly unstable as a solution of the vacuum Einstein equation (1.7), and the instability can be realized as a mode instability in harmonic/transverse-traceless gauge (1.10), which is not pure gauge.*

Remark. One can construct a gauge invariant quantity, the tztz-component of the linearized Weyl tensor $\overset{(1)}{W}$, *which is non-vanishing for a non-trivial mode solution h with* $k \neq 0$ *and* $\mu \neq 0$ *and exhibits exponential growth in t when* $\mu > 0$. *This allows one to show that the mode solution constructed in Theorem 1.1 is not pure gauge. Hence, one expects that the above mode solution persists in any "good" gauge, not just (1.10).*

Remark. The reader should note that the lower bound on the frequency parameter k should not be interpreted as ruling out the existence of unstable modes with arbitrarily long wavelengths. The lower bound on k in Theorem 1.1 results from the use of a test function in the variational argument (see Proposition 4.5 in Sec. IV C). The numerics of Gregory–Laflamme and Hovdebo–Myers[1,8] *both provide evidence that there are unstable modes for k arbitrarily small.*

D. Difficulties and main ideas of the proof

It may seem natural to directly consider the problem in harmonic gauge since the equation of study (1.7) reduces to a tensorial wave equation

$$g^{cd}\nabla_c\nabla_d h_{ab} + 2R_a{}^c{}_b{}^d h_{cd} = 0. \tag{1.11}$$

The above equation (1.11) results from the linearization of the gauge reduced non-linear vacuum Einstein equation (1.1), which is strongly hyperbolic and therefore well-posed. Equation (1.11) reduces to a system of ordinary differential equations (ODEs) under the mode solution ansatz (1.8) with (1.9). This system can be reduced to a single ODE of the form

$$\frac{d^2u}{dr^2} + P_{\mu,k}(r)\frac{du}{dr} + Q_{\mu,k}(r)u = \frac{\mu^2}{D(r)^2}u, \qquad D = 1 - \frac{2M}{r}, \tag{1.12}$$

where $u = H_{tt}$, H_{tr}, H_{rr}, or $H_{\theta\theta}$, and $P_{\mu,k}(r)$ and $Q_{\mu,k}(r)$ depend on μ, k, and r. However, if one insists on this decoupling, one introduces a regular singular point in the range $r \in (0,\infty)$. For certain ranges of μ and k, this value occurs on the exterior $\mathcal{E}_A$, i.e., the regular singular point occurs in $r \in (2M,\infty)$. In particular, this regular singularity occurs on the exterior for the numerical values of k and μ for which Gregory–Laflamme identified instability. In the original works of Gregory and Laflamme, the decoupled ODE for H_{tr} was studied (see Refs. 1, 7, and 9)

It turns out that in looking for an instability, one can make a different gauge choice called spherical gauge. As shown in Sec. III, the linearized vacuum Einstein equation (1.7) for a mode solution (1.8) in spherical gauge can be reduced to a second-order ODE of the form (1.12), where, in contrast to harmonic/transverse-traceless gauge, $P_{\mu,k}(r) = P_k(r)$ and $Q_{\mu,k}(r) = Q_k(r)$ depend only on k and r. Hence, the existence of solution to ODE (1.12) becomes a simple eigenvalue problem for μ. Spherical gauge was originally introduced in Ref. 18 and has another advantage over harmonic/transverse-traceless gauge, which is that all $r \in (2M, \infty)$ are ordinary points of ODE (1.12). Hence, the spherical gauge choice also avoids the issues of a regular singularity at some $r \in (2M, \infty)$. However, in contrast to harmonic gauge, for this gauge choice, well-posedness is unclear. If one were trying to prove *stability,* then exhibiting a well-posed gauge would be key since well-posedness of the equations is essential for understanding general solutions. For *instability*, it turns out that it is sufficient to exhibit a mode solution of the non-gauge reduced Eq. (1.7), which is *not pure gauge*. One expects then that such a mode solution will persist in all "good" gauges, of which harmonic gauge is an example. The discussion of pure gauge mode solutions in spherical gauge in Sec. III C provides a proof that if $k \neq 0$ and $\mu \neq 0$, then a mode solution in spherical gauge is *not* pure gauge. This can be shown directly or from the computation of a gauge invariant quantity, namely, the *tztz*-component of the linearized Weyl tensor, $\overset{(1)}{W}$. Furthermore, it is shown that if a non-trivial mode solution in spherical gauge grows exponentially in t, then $\overset{(1)}{W}_{tztz}$ is non-zero and grows exponentially t.

An issue with spherical gauge is that mode solutions in the spherical gauge do not, in general, extend smoothly to the future event horizon $\mathcal{H}_A^+$, even when they represent physically admissible solutions. However, as shown in Sec. III D, one can detect what are the admissible boundary conditions at the future event horizon in spherical gauge by adding a pure gauge perturbation to the metric perturbation to try and construct a solution that indeed extends smoothly to $\mathcal{H}_A^+$. In fact, the pure gauge perturbation found is precisely one that transforms the metric perturbation to harmonic/transverse-traceless gauge (1.10). Hence, after also identifying the admissible boundary conditions at spacelike infinity i_A^0 in Sec. III D, proving the existence of an unstable mode solution to the linearized vacuum Einstein equation (1.7) that is *not* pure gauge is reduced to showing the existence of a solution to ODE (1.12) with $\mu > 0$ and $k \neq 0$, which satisfies the admissible boundary conditions that are identified in this work.

In this paper, ODE problem (1.12) is approached from a direct variational point of view in Sec. IV. To run a direct variational argument, the solution u of ODE (1.12) is rescaled and change of coordinates is applied. It is shown in Sec. IV A that Eq. (1.12) can be cast into a Schrödinger form

$$-\Delta_{r_*} u + V_k(r_*)u = -\mu^2 u, \qquad r_* = r + 2M\log(r - 2M), \tag{1.13}$$

with V_k independent of μ. ODE (1.13) can be interpreted as an eigenvalue problem for $-\mu^2$; finding an eigenfunction, in a suitable space, with a negative eigenvalue will correspond to an instability. As shown in Sec. IV B, this involves assigning the following energy functional to the Schrödinger operator on the left-hand side of (1.13):

$$E(u) := \langle \nabla_{r_*} u, \nabla_{r_*} u \rangle_{L^2(\mathbb{R})} + \langle V_k u, u \rangle_{L^2(\mathbb{R})}. \tag{1.14}$$

Using a suitably chosen test function, one can show that the infimum over functions in $H^1(\mathbb{R})$ of this functional is negative for a range of k. One then needs to argue that this infimum is attained as an eigenvalue by showing that this functional is lower semicontinuous and that the minimizer is non-trivial. The corresponding eigenfunction is then a weak solution in $H^1(\mathbb{R})$ to ODE (1.13) with $\mu > 0$ for a range of $k \in \mathbb{R}\backslash\{0\}$. Elementary one-dimensional elliptic regularity implies that the solution is indeed smooth away from the future event horizon, $\mathcal{H}_A^+$, and therefore corresponds to a classical solution of the problem (1.13). Finally, the solution can be shown to satisfy the admissible boundary conditions by the condition that the solution lies in $H^1(\mathbb{R})$.

This paper is organized in the following manner. The remainder of the present section contains additional background on the Gregory–Laflamme instability. In Sec. II, linear perturbation theory is reviewed and the linearized Einstein equation (1.7) is derived. In Sec. III, the analysis in spherical gauge is presented. The decoupled ODE (1.12) resulting from the linearized Einstein equation (1.7) is derived, and it is established that the problem can be reduced to the existence of a solution to the decoupled ODE with $\mu > 0$ and $k \neq 0$ satisfying admissible boundary conditions. In Sec. IV, the proof of the existence of such a solution is presented via the direct variational method.

Appendix A contains a list of the Riemann tensor components and the Christoffel symbols for the Schwarzschild black string spacetime $\mathrm{Sch}_4 \times \mathbb{R}$ or $\mathrm{Sch}_4 \times \mathbb{S}^1_R$. Appendix B collects results on singularities in second order ODE relevant for the discussion of the boundary conditions for the decoupled ODE (1.12). Appendix C provides a method of transforming a second order ODE into a Schrödinger equation. Appendix D collects some useful results from analysis that are needed in the Proof of Theorem 1.1. Appendix E compliments Theorem 1.1 with some stability results.

E. The canonical energy method

The reader should note that there are two papers[5,6] concerning a very general class of spacetimes, which are of relevence to the stability problem for the Schwarzschild black string. In particular, it follows from Refs. 5 and 6 that there exists a linear perturbation of the Schwarzschild black string spacetime, which is not pure gauge and grows exponentially in the Schwarzschild t-coordinate. The following describes the results of these works.

In 2012, a paper of Hollands and Wald[5] gave a criterion for linear stability of stationary, axisymmetric, vacuum black holes and black branes in $D \geq 4$ spacetime dimensions under axisymmetric perturbations. They define a quantity called the "canonical energy" $\mathcal{E}$ of the perturbation, which is an integral over an initial Cauchy surface of an expression quadratic in the perturbation. It can be related to thermodynamic quantities by

$$\mathcal{E} = \delta^2 M - \sum_B \Omega_B \delta^2 J_B - \frac{\kappa}{8\pi}\delta^2 A, \tag{1.15}$$

where M and J_B are the ADM mass and ADM angular momenta in the B^{th} plane and A is the cross-sectional area of the horizon. Note that the right-hand side of (1.15) refers to the second variation of thermodynamic quantities. It is remarkable that the combination $\mathcal{E}$ of these second variations is, in fact, determined by linear perturbations.

Reference 5 considers initial data for a perturbation of either a stationary, axisymmetric black hole or black brane with the following properties: (i) the linearized Hamiltonian and momentum constraints are satisfied, (ii) that $\delta M = 0 = \delta J_A$ and that the ADM momentum vanishes, and (iii) specific gauge conditions and finiteness/regularity conditions at the future horizon and infinity are satisfied. In what follows, initial data satisfying (i)–(iii) will be referred to as admissible. Hollands and Wald showed that if $\mathcal{E} \geq 0$ for all admissible initial data, then one has mode stability. The work also establishes that if there exist admissible initial data such that $\mathcal{E} < 0$, then there exist admissible initial data for a perturbation, which cannot approach a stationary perturbation at late times, i.e., one has failure of asymptotic stability.

For the Schwarzschild black hole, one can take initial data, which corresponds simply to a change of the mass parameter $M \mapsto M + \alpha$, and therefore, by Eq. (1.15) and since the cross-sectional area of the horizon is given by $A = 16\pi(M+\alpha)^2$, it follows that $\mathcal{E} < 0$. This is the "thermodynamic instability" of the Schwarzschild black hole. However, the initial data for a change of mass perturbation is manifestly not admissible (the family of Schwarzschild black holes is, after all, dynamically stable).

The work of Hollands and Wald[5] also shows an additional result relevant specifically to the problem of stability of black *branes*. Suppose that there exist initial data for a perturbation of the ADM parameters of a black *hole* such that $\mathcal{E} < 0$. Reference 5 shows that starting from such a perturbation of the black *hole*, one can infer the existence of admissible initial data, which depend on a parameter l, for a perturbation (which is not pure gauge) of the associated black *brane* such that again $\mathcal{E} < 0$. One should note that this argument does not give an explicit bound on l. This criterion formalized a conjecture by Gubser–Mitra that a necessary and sufficient condition for stability of the black brane spacetimes is thermodynamic stability of the corresponding black hole.[12,13] Since the change of mass perturbation of the Schwarzschild black hole produces $\mathcal{E} < 0$, this argument implies that the Schwarzschild black string fails to be asymptotically stable.

Remark. The reader should note that the Hollands and Wald paper[5] also showed that a necessary and sufficient condition for stability, with respect to axisymmetric perturbations, is that a "local Penrose inequality" is satisfied. The idea that a local Penrose inequality gives a stability criterion was originally discussed in the work of Figueras, Murata, and Reall,[4] which gave strong evidence in favor of sufficiency of this condition for stability. Furthermore, Ref. 4 showed numerically that this local Penrose inequality was violated for the Schwarzschild black string for a range of frequency parameters that closely match those found in the original work of Gregory–Laflamme.[1]

The failure of asymptotic stability does not in itself imply that perturbations grow. However, the results of Ref. 5 were strengthened in 2015 by Prabhu and Wald.[6] They showed, using some spectral theory, that if there exist admissible initial data for a perturbation such that $\mathcal{E} < 0$ for a black *brane*, then there exists initially well-behaved perturbations that are not pure gauge and that grow exponentially in time. Having established that there exist admissible initial data for a perturbation such that $\mathcal{E} < 0$ for the Schwarzschild black string in Ref. 5, the existence of a linear perturbation, which is not pure gauge and has exponential growth, follows.

The present work differs from the above as it gives a direct, self-contained, elementary proof of the Gregory–Laflamme instability following the original formulation of Refs. 1 and 7–9, which is completely explicit. In particular, it gives an exponentially growing mode solution with an explicit growth rate of the form defined by equations (1.8) and (1.9) in harmonic/transverse-traceless gauge, which is not pure gauge.

Remark. It would also be of interest to see if Theorem 1.1 in the form stated could be inferred from the canonical energy method of Hollands, Wald, and Prabu[5,6] in an explicit way bypassing some of the functional calculus applied there. In particular, it would be interesting to explore the possible relation between the variational theory applied to $\mathcal{E}$ and that applied here (see Sec. IV B).

F. Outlook

This paper brings together what is known about the Gregory–Laflamme instability as well as providing a direct elementary mathematically rigorous proof of its existence without the use of numerics and with an explicit bound on μ and k. Note that while only the 5D Schwarzschild black string was considered here, the result of instability readily extends to higher dimensions with the replacement of kz in the exponential factor with $\sum_i k_i z_i$.

Further directions of work could be to study the non-linear problem, the extension to $\text{Kerr}_4 \times \mathbb{S}^1$ or $\text{Kerr}_4 \times \mathbb{R}$, the extension to charged black branes of the work,[10] and the extension to black rings or ultraspinning Myers–Perry black holes.

G. Contextual remarks

1. Motivation for the study of higher dimensions

The study of higher dimensions merits a few words of motivation since, from a physical standpoint, only 3 + 1 are perceived classically. First, from a purely mathematical perspective, it is of interest to see how general relativity differs in higher dimensions from the 4*D* case. This throws light on how general Lorentzian manifolds obeying the vacuum Einstein equation (1.1) behave. Second, the physics community is very interested in higher-dimensional gravity from the point of view of string theory. Understanding how general relativity behaves in higher dimensions is therefore of relevance to the low energy limit of string theory.[2]

2. Some differences in higher dimensions

In higher dimensions, many results from 4*D* general relativity no longer hold. As shown by Hawking, in 4*D*, the cross sections of the event horizon of an asymptotically flat stationary black hole spacetime must be topologically $\mathbb{S}^2$ (under the dominant energy condition).[22] In higher dimensions, it is possible to construct explicit examples of black hole spacetimes with non-spherical cross-sectional horizon topology. For example, the black ring solution with horizon topology is $\mathbb{S}^2 \times \mathbb{S}^1$.[19] In higher dimensions, there also exists a generalized Kerr solution known as the Myers–Perry black hole,[17] which has cross-sectional horizon topology $\mathbb{S}^3$. Hawking's theorem has been generalized to higher dimensions,[23] which shows that the horizon topology must be of positive scalar curvature. In 5D, under the assumptions of stationarity, asymptotic flatness, two commuting axisymmetries and "rod structure" black holes are unique, and further the horizon topology is either $\mathbb{S}^3$, $\mathbb{S}^1 \times \mathbb{S}^2$, or lens space.[24]

In 4*D*, it is conjectured that maximal developments of "generic" asymptotically flat initial data sets can asymptotically be described by a finite number of Kerr black holes. This "final state conjecture" cannot generalize immediately since there exist at least two distinct families of black hole solutions that can have the same mass and angular momentum: the Myers–Perry black hole and the black ring. Moreover, there exist distinct black ring solutions with the same mass and angular momentum.[2,20] The final state conjecture may need to be modified to include the property of stability.

3. Related works

A few other works are of relevance to this discussion. The review paper[2] and book chapter[20] discuss the black ring solution[19] in great detail. This relates to the work presented here since the Gregory–Laflamme instability is often heuristically invoked when discussing higher-dimensional black hole solutions. In particular, if the black ring of study has a large radius and is sufficiently thin, then it "looks like" a Schwarzschild black string and therefore would be susceptible to the Gregory–Laflamme instability. There have been heuristic and numerical results to give evidence to this claim.[8,25] Finally, note that in 2018, Ref. 26 produced the first mathematically rigorous result on the stability problem for the black ring spacetime.

II. LINEAR PERTURBATION THEORY

This section provides a derivation and review of the linearized vacuum Einstein equation (1.7) around a general spacetime background metric (M, g) satisfying the vacuum Einstein equation (1.1).

A. Linearized vacuum Einstein equation

Consider a Lorentzian manifold (M, g) with metric satisfying the vacuum Einstein equation

$$\mathrm{Ric}_g = 0. \tag{2.1}$$

In this section, a "perturbation" of the spacetime metric will be discussed. This will be represented by a new metric of the form $g + \epsilon h$ with $\epsilon > 0$. h here is a symmetric bilinear form on the fibers of TM. In the following, a series of results on how various quantities change to $O(\epsilon)$ (the linear level) are derived. This will result in an expression for the Ricci tensor under such a perturbation to linear order.

Remark. An important point to note is that indices are raised and lowered here with respect to g.

Proposition 2.1 (change in the Ricci Tensor). Consider a Lorentzian manifold (M, g). Suppose the metric $\tilde{g}_{ab} = g_{ab} + \epsilon h_{ab}$ is a Lorentzian metric. Then, the Ricci tensor, $(\widetilde{\mathrm{Ric}_g})_{ab}$, of $\tilde{g}_{ab}$ to $O(\epsilon)$ is

$$(\widetilde{\mathrm{Ric}_g})_{ab} = (\mathrm{Ric}_g)_{ab} - \epsilon \frac{1}{2} \Delta_L h_{ab}, \tag{2.2}$$

where Δ_L denotes the Lichnerowicz operator given by

$$\Delta_L h_{ab} = g^{cd} \nabla_c \nabla_d h_{ab} + 2R_a{}^c{}_b{}^d h_{cd} - 2(\mathrm{Ric}_g)_{c(a} h_{b)}{}^c - 2\nabla_{(a} \nabla^c h_{b)c} + \nabla_a \nabla_b h, \tag{2.3}$$

and $h = g^{ab}h_{ab}$.

Proof. Direct computation. □

If one assumes that g satisfies the vacuum Einstein equation (1.1) and $g + \epsilon h$ satisfies the vacuum Einstein equation (1.1) to $O(\epsilon)$, then it follows from Proposition 2.1 that h must satisfy

$$g^{cd}\nabla_c\nabla_d h_{ab} + \nabla_a\nabla_b h - 2\nabla_{(b}\nabla^c h_{a)c} + 2R_a{}^c{}_b{}^d h_{cd} = 0 \tag{2.4}$$

to $O(\epsilon)$. In what follows, Eq. (2.4) will be called the linearized vacuum Einstein equation. This will be the main equation of interest, with g being the Schwarzschild black string metric,

$$g := -D(r)dt\otimes dt + \frac{1}{D(r)}dr\otimes dr + r^2\left(d\theta\otimes d\theta + \sin^2\theta d\varphi\otimes d\varphi\right) + dz\otimes dz, \quad D(r) = 1 - \frac{2M}{r}. \tag{2.5}$$

B. Pure gauge solutions in linearized theory

The vacuum Einstein equation (1.1) is a system of second order quasilinear partial differential equations of the pair (M, g), which are invariant under the diffeomorphisms of M. This means that for given initial data, the vacuum Einstein equation (1.1) only determines a spacetime unique up to diffeomorphism, i.e., if there exists a diffeomorphism $\Phi : M \to M$, then (M, g) and $(M, \Phi_*(g))$ are equivalent solutions of the vacuum Einstein equation (1.1). For constructing spacetimes, one often imposes conditions on local coordinates called a gauge choice. For linearized theory, this can be formulated as follows.

Consider a Lorentzian manifold $(M, \tilde{g} := g + \epsilon h)$ with $\epsilon > 0$. Let $\{\Phi_\tau\}$ be a one-parameter family of diffeomorphisms generated by a vector field X and define $\xi := \tau X \in TM$. Then, from the definition of the Lie derivative, one has

$$(\Phi_\tau)_*(\tilde{g}) = \tilde{g} + \mathcal{L}_\xi g + \mathcal{O}(\epsilon^2) \tag{2.6}$$

if one treats $\tau = \mathcal{O}(\epsilon)$. Hence, in the context of linearized theory, one considers two solutions to the linearized vacuum Einstein equation (2.4), h_1 and h_2, as equivalent if

$$h_2 = h_1 + \mathcal{L}_\xi g \Longleftrightarrow (h_2)_{ab} = (h_1)_{ab} + 2\nabla_{(a}\xi_{b)} \tag{2.7}$$

for some vector field $\xi \in TM$.

Definition 2.1 (Pure Gauge Solution). Let (M, g) be a vacuum spacetime. A solution h to the linearized vacuum Einstein equation (2.4) will be called pure gauge if there exists a vector field $\xi \in TM$ such that

$$h_{ab} = 2\nabla_{(a}\xi_{b)}. \tag{2.8}$$

The notation h_{pg} will be used to denote a pure gauge solution to the linearized vacuum Einstein equation (2.4).

Showing that a solution h to the linearized vacuum Einstein equation (2.4) is *not* pure gauge is tantamount to showing that h is not equivalent to the trivial solution. It is thus essential that the solution constructed in this paper *not* be pure gauge. The following propositions establish that the *tztz*-component of the linearized Weyl tensor $\overset{(1)}{W}$ is invariant under gauge transformation. This means that if $\overset{(1)}{W}$ is non-zero for a solution h to the linearized vacuum Einstein equation (2.4), then h cannot be pure gauge.

Proposition 2.2 (Change to the Weyl Tensor). Let (M, g) be a vacuum spacetime. Suppose the metric $\tilde{g}_{ab} = g_{ab} + \epsilon h_{ab}$ is a Lorentzian metric such that h satisfies the linearized vacuum Einstein equation (2.4). Then the Weyl tensor, $\tilde{W}_{abcd}$, of $\tilde{g}_{ab}$ to $O(\epsilon)$ is

$$\tilde{W}_{abcd} = W_{abcd} + \epsilon \overset{(1)}{W}_{abcd}, \tag{2.9}$$

where

$$\overset{(1)}{W}_{abcd} = \nabla_c\nabla_{[b}h_{a]d} + \nabla_d\nabla_{[a}h_{b]c} + \frac{1}{2}\left(R^e{}_{bcd}h_{ae} - R^e{}_{acd}h_{eb}\right). \tag{2.10}$$

Henceforth, $\overset{(1)}{W}$ will be referred to as the linearized Weyl tensor.

Proof. Direct computation. □

Proposition 2.3. For the 5D Schwarszchild black string, $\overset{(1)}{W}_{tztz}$ evaluated on a pure gauge solution vanishes.

Proof. Let $\overset{(1)}{W}_{\mathrm{pg}}$ denote the linearized Weyl tensor evaluated on a pure gauge solution h_{pg}. Recall that a pure gauge solution h_{pg} can always be written as $h_{\mathrm{pg}} = \mathcal{L}_\xi g$ for some vector field $\xi \in TM$. Using Proposition 2.2, one has that

$$(\overset{(1)}{W}_{\mathrm{pg}})_{abcd} = \nabla_c\nabla_{[b}\nabla_{a]}\xi_d + \nabla_d\nabla_{[a}\nabla_{b]}\xi_c + \frac{1}{2}\left(R^e{}_{bcd}\nabla_a\xi_e - R^e{}_{acd}\nabla_e\xi_b\right) \\ + \nabla_{[c}\nabla_{|b|}\nabla_{d]}\xi_a + \nabla_{[d}\nabla_{|a|}\nabla_{c]}\xi_b + \frac{1}{2}\left(R^e{}_{bcd}\nabla_e\xi_a - R^e{}_{acd}\nabla_b\xi_e\right). \tag{2.11}$$

By repeated use of the Ricci identity with the first and second Bianchi identities, one can compute that

$$(\overset{(1)}{W}_{\mathrm{pg}})_{abcd} = (\nabla_a R)^e{}_{bcd}\xi_e + (\nabla_b R)^e{}_{adc}\xi_e + R^e{}_{adc}\nabla_b\xi_e + R^e{}_{bcd}\nabla_a\xi_e + R^e{}_{dab}\nabla_c\xi_e + R^e{}_{cba}\nabla_d\xi_e. \tag{2.12}$$

From Appendix A, one has $R^\mu{}_{\alpha\beta z} = 0$, $R^\mu{}_{\alpha z\beta} = 0$, $R^\mu{}_{z\alpha\beta} = 0$, and $\Gamma^\alpha_{z\beta} = 0$. Furthermore, the black string metric (1.4) is independent of t and z. Hence,

$$(\overset{(1)}{W}_{\mathrm{pg}})_{tztz} = 0. \tag{2.13}$$

□

III. ANALYSIS IN SPHERICAL GAUGE

In this section, a mode solution, h, of the linearized vacuum Einstein equation (2.4) on the exterior $\mathcal{E}_A$ of the Schwarzschild black string spacetime $\mathrm{Sch}_4 \times \mathbb{R}$ or $\mathrm{Sch}_4 \times \mathbb{S}^1_R$ is considered. One makes the additional assumption that this mode solution preserves the spherical symmetry of Sch_4. Hence, in particular, the solution can be expressed in $(t, r, \theta, \varphi, z)$ coordinates as

$$h_{\alpha\beta} = e^{\mu t + ikz}\begin{pmatrix} H_{tt}(r) & H_{tr}(r) & 0 & 0 & H_{tz}(r) \\ H_{tr}(r) & H_{rr}(r) & 0 & 0 & H_{rz}(r) \\ 0 & 0 & H_{\theta\theta}(r) & 0 & 0 \\ 0 & 0 & 0 & H_{\theta\theta}(r)\sin^2\theta & 0 \\ H_{tz}(r) & H_{rz}(r) & 0 & 0 & H_{zz}(r)\end{pmatrix}, \tag{3.1}$$

where $\alpha, \beta \in \{t, r, \theta, \varphi, z\}$. Moreover, in search of instability, the most interesting case for the present work is $\mu > 0$.

This section contains the analysis of the ODEs resulting from the linearized Einstein vacuum equation (2.4) for a mode solution of the form (3.1) when it is expressed in spherical gauge.

Definition 3.1 (Spherical Gauge). A mode solution h of the linearized vacuum Einstein equation (2.4) on the exterior $\mathcal{E}_A$ of the Schwarzschild black string spacetime $\mathrm{Sch}_4 \times \mathbb{R}$ *is said to be in spherical gauge if it is of the form*

$$h_{\mu\nu} = e^{\mu t + ikz}\begin{pmatrix} H_t(r) & \mu H_v(r) & 0 & 0 & 0 \\ \mu H_v(r) & H_r(r) & 0 & 0 & -ikH_v(r) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -ikH_v(r) & 0 & 0 & H_z(r)\end{pmatrix}. \tag{3.2}$$

For the Schwarzschild black string spacetime $\mathrm{Sch}_4 \times \mathbb{S}^1_R$, *one makes the same definition with the additional assumption that* $kR \in \mathbb{Z}$.

Remark. The terminology "spherical gauge" is motivated by the fact that a mode solution of this form preserves the area of the spheres of the original spacetime.

First, it is shown in Sec. III A that one can impose the gauge consistently at the level of modes, i.e., if there is a mode solution of the form (3.1), with $\mu \neq 0$ and either $k \neq 0$ or $\frac{dH_{tz}}{dr} - H_{rz} = 0$, then there is a mode solution of the form (3.2) differing from the original one by a pure gauge solution. In the case where $H_{tz} = 0$, $H_{rz} = 0$, and $H_{zz} = 0$, this consistency condition is already implicit in Refs. 18 and 8. In Sec. III B, the original decoupling of the ODEs resulting from the linearized vacuum Einstein equation (2.4) and the spherical gauge ansatz (3.2) is reproduced from Ref. 8. This decoupling results in a single ODE for the component $H_z(r)$ in Eq. (3.2). It is then shown, in Sec. III C, that if $k \neq 0$ and $\mu \neq 0$, then mode solutions in spherical gauge (3.2) are not pure gauge. This is proved by examining the $tztz$-component of the linearized Weyl tensor $\overset{(1)}{W}$ associated with a mode solution in spherical gauge, which is gauge invariant by Proposition 2.3. In this section, it is also proved that if a non-trivial mode solution in spherical gauge has $\mu > 0$ (i.e., it grows exponentially in t) and $k \neq 0$, then $\overset{(1)}{W}_{tztz}$ is non-zero and also grows exponentially. By the gauge invariance of $\overset{(1)}{W}_{tztz}$, this behavior will persist in all gauges. Next, in Sec. III D, the admissible boundary conditions for the solution at the future event horizon $\mathcal{H}_A^+$ and finiteness conditions at spacelike infinity i_A^0 are identified. Note this issue is subtle since, in general, both "basis" elements for a mode solution h of the form (3.2) are, in fact, singular at the future event horizon $\mathcal{H}_A^+$ in this gauge. By adding a pure gauge perturbation, the admissible boundary conditions for the solution h in the form (3.2) can be identified. Moreover, this pure gauge solution can be chosen such that, after adding it, the harmonic/transverse-traceless gauge (1.10) conditions are satisfied. Finally, in Sec. III E, the problem of constructing a linear mode instability of the form (3.1) is reduced to show that there exists a solution to the decoupled ODE for $H_z(r)$, with $\mu > 0$ and $k \neq 0$, that satisfies the admissible boundary conditions at the future event horizon $\mathcal{H}_A^+$ and spacelike infinity i_A^0 (see Proposition 3.8).

A. Consistency

In Ref. 8, it is stated that any mode solution of the form in Eq. (3.1) with $H_{tz} = 0$, $H_{rz} = 0$, and $H_{zz} = 0$ can be brought to the spherical gauge form (3.2) by the addition of a pure gauge solution. Slightly more generally, one, in fact, has the following proposition:

Proposition 3.1 (Consistency of the Spherical Gauge). Consider a mode solution h to the linearized Einstein vacuum equation (2.4) on the exterior $\mathcal{E}_A$ of the Schwarzschild black string spacetime $\mathrm{Sch}_4 \times \mathbb{R}$ *or* $\mathrm{Sch}_4 \times \mathbb{S}^1_R$ *of the form (3.1) with $\mu \neq 0$. Further suppose that either $k \neq 0$ or $\frac{d}{dr}H_{tz} - \mu H_{rz} = 0$. Then, there exists a pure gauge solution h_{pg} such that $h + h_{\mathrm{pg}}$ is of form (3.2). It is in this sense that the spherical gauge (3.2) can be consistently imposed on the exterior $\mathcal{E}_A$ of the Schwarzschild black string* $\mathrm{Sch}_4 \times \mathbb{R}$ *or* $\mathrm{Sch}_4 \times \mathbb{S}^1_R$.

Proof. From Sec. II B, a pure gauge solution is given by $h_{\mathrm{pg}} = 2\nabla_{(a}\xi_{b)}$ for a vector field ξ. Hence, $\tilde{h}_{ab} = h_{ab} + 2\nabla_{(a}\xi_{b)}$ is the new mode solution. Consider a diffeomorphism generating vector field of the form $\xi = e^{\mu t + ikz}(\zeta_t(r), \zeta_r(r), 0, 0, \zeta_z(r))$.

If $k \neq 0$, one can take

$$\begin{aligned} \zeta_t(r) &= \frac{ir(r-2M)}{2Mk}(\partial_r H_{tz}(r) - \mu H_{rz}(r)) + \frac{r(r-2M)}{2M}H_{tr}(r) - \frac{r\mu}{2M}H_{\theta\theta}(r), \\ \zeta_r(r) &= -\frac{H_{\theta\theta}(r)}{2(r-2M)}, \\ \zeta_z(r) &= -\frac{(H_{tz}(r) + ik\zeta_t(r))}{\mu} \end{aligned} \tag{3.3}$$

and immediately verify that $\tilde{h}$ is of the form (3.2).

If $\frac{d}{dr}H_{tz} - \mu H_{rz} = 0$, then one can take

$$\zeta_t(r) = \frac{r(r-2M)}{2M}H_{tr}(r) - \frac{r\mu}{2M}H_{\theta\theta}(r), \quad \zeta_r(r) = -\frac{H_{\theta\theta}(r)}{2(r-2M)}, \quad \zeta_z(r) = -\frac{(H_{tz}(r) + ik\zeta_t(r))}{\mu} \tag{3.4}$$

and immediately verify that $\tilde{h}$ is of the form (3.2).

□

B. Reduction to ODE

Under a spherical gauge ansatz (3.2) with $\mu \neq 0$ and $k \neq 0$, the linearized vacuum Einstein equation (2.4) reduces to a system of coupled ODEs for the components H_t, H_v, H_r, and H_z. This system of ODEs can be decoupled to the single ODE for $\mathfrak{h} := H_z$,

$$\frac{d^2\mathfrak{h}}{dr^2}(r) + P_k(r)\frac{d\mathfrak{h}}{dr}(r) + \left(Q_k(r) - \frac{\mu^2 r^2}{(r-2M)^2}\right)\mathfrak{h}(r) = 0, \tag{3.5}$$

with

$$P_k(r) := \frac{12M}{r(k^2r^3 + 2M)} - \frac{5}{r} + \frac{1}{r-2M}, \tag{3.6}$$

$$Q_k(r) := \frac{6M}{r^2(r-2M)} - \frac{rk^2}{r-2M} - \frac{12M^2}{r^2(r-2M)(k^2r^3+2M)}. \tag{3.7}$$

The following proposition establishes this decoupling of the linearized vacuum Einstein equation (2.4) to ODE (3.5) and the construction of a mode solution h in spherical gauge (3.2) from a solution $\mathfrak{h}$ to ODE (3.5).

Proposition 3.2. Given a mode solution h in spherical gauge (3.2) with $\mu \neq 0$ and $k \neq 0$ on the exterior $\mathcal{E}_A$ of the Schwarzschild black string $\mathrm{Sch}_4 \times \mathbb{R}$ or $\mathrm{Sch}_4 \times \mathbb{S}^1_R$, *ODE (3.5) is satisfied by h_{zz}. Conversely, given a $C^2((2M,\infty))$ solution $\mathfrak{h}(r)$ to ODE (3.5) with $k \neq 0$ and $\mu \neq 0$, one can* *construct a mode solution h in spherical gauge (3.2) to the linearized vacuuum Einstein equation (2.4) on the exterior $\mathcal{E}_A$ of the Schwarzschild black string* $\mathrm{Sch}_4 \times \mathbb{R}$. *If $kR \in \mathbb{Z}$, then h induces a mode solution on* $\mathrm{Sch}_4 \times \mathbb{S}^1_R$.

Proof. Let h be a mode solution in spherical gauge (3.2) with $\mu \in \mathbb{R}$ and $k \in \mathbb{R}$ satisfying the linearized vacuum Einstein equation (2.4) on the exterior $\mathcal{E}_A$ of the Schwarzschild black string $\mathrm{Sch}_4 \times \mathbb{R}$ *or* $\mathrm{Sch}_4 \times \mathbb{S}^1_R$. Equivalently, the following system of ODE has to be satisfied:

$$\mu k H_r = \frac{2M\mu k}{r(r-2M)} H_v, \tag{3.8}$$

$$\mu k^2 H_v = \frac{\mu}{2}\frac{dH_z}{dr} - \frac{\mu(r-2M)H_r}{r^2} - \frac{\mu M H_z}{2r(r-2M)}, \tag{3.9}$$

$$k\frac{dH_t}{dr} = \frac{kMH_t}{r(r-2M)} - \frac{k(r-2M)(2r-3M)H_r}{r^3} + 2\,\mu^2 k H_v, \tag{3.10}$$

$$\begin{aligned} H_t = {}& \frac{(r-2M)(r(k^2-\mu^2)-2Mk^2)}{M} H_v + \frac{(r-2M)^2(r+M)}{Mr^2} H_r, \\ & + \frac{(r-2M)^3}{2Mr}\frac{dH_r}{dr} - \frac{(r-2M)^2}{2M}\frac{dH_z}{dr} + \frac{r(r-2M)}{2M}\frac{dH_t}{dr}, \end{aligned} \tag{3.11}$$

$$\frac{d^2H_z}{dr^2} = k^2 H_r + \frac{r^2}{(r-2M)^2}\left(\mu^2 H_z - k^2 H_t\right) + \frac{2(r-M)}{r(r-2M)}\left(2k^2 H_v - \frac{dH_z}{dr}\right) + 2k^2\frac{dH_v}{dr}, \tag{3.12}$$

$$\begin{aligned} \frac{d^2H_z}{dr^2} = {}& \frac{2M(2r-3M)}{r(r-2M)^3} H_t - \frac{\left(6M^2 - (\mu^2+k^2)r^4 + 2Mr(k^2r^2-2)\right)}{r^3(r-2M)} H_r, \\ & - \frac{2M(2Mk^2 + r(\mu^2-k^2))}{r(r-2M)^2} H_v + \frac{2r-3M}{r^2}\frac{dH_r}{dr} - \frac{2\mu^2 r + 4Mk^2 - 2k^2 r}{r-2M}\frac{dH_v}{dr} \\ & - \frac{M}{r(r-2M)}\frac{dH_z}{dr} - \frac{M}{(r-2M)^2}\frac{dH_t}{dr} + \frac{r}{r-2M}\frac{d^2H_t}{dr^2}, \end{aligned} \tag{3.13}$$

$$\begin{aligned} \frac{d^2H_t}{dr^2} = {}& \frac{k^2r^4 - 2Mk^2r^3 - 2M^2}{r^2(r-2M)^2} H_t - \left(\mu^2 + \frac{2M^2}{r^4}\right) H_r - \frac{r\mu^2}{r-2M} H_z \\ & + \frac{4\mu^2r^2 + 4M^2k^2 - 2Mr(3\mu^2+k^2)}{r^2(r-2M)} H_v - \frac{2r-5M}{r(r-2M)}\frac{dH_t}{dr} - \frac{M(r-2M)}{r^3}\frac{dH_r}{dr} \\ & + 2\mu^2\frac{dH_v}{dr} + \frac{M}{r^2}\frac{dH_z}{dr}. \end{aligned} \tag{3.14}$$

Now, if $\mu \neq 0$ and $k \neq 0$, then from Eqs. (3.8) and (3.9), one can find H_v in terms of H_z and $\frac{dH_z}{dr}$. This can then be used in Eq. (3.10) to give an equation for $\frac{dH_t}{dr}$ in terms of H_t, H_z, and $\frac{dH_z}{dr}$. All of these expressions can be used to express H_t in terms of H_z, $\frac{dH_z}{dr}$, and $\frac{d^2H_z}{dr^2}$ via Eq. (3.11). The resulting equations are

$$H_r(r) = -\frac{M^2 r}{(r-2M)^2(k^2r^2+2M)} H_z(r) + \frac{Mr^2}{(r-2M)(k^2r^2+2M)}\frac{dH_z}{dr}, \tag{3.15}$$

$$H_v(r) = -\frac{Mr^2}{(2(r-2M)(k^2r^2+2M)} H_z(r) + \frac{r^3}{2(k^2r^2+2M)}\frac{dH_z}{dr}, \tag{3.16}$$

$$\begin{aligned} H_t(r) = {}& \frac{2M^2(r-3M) + Mk^2r^3(2r-5M) - k^4r^6(r-2M)}{r(k^2r^3+2M)^2} H_z, \\ & - \frac{2(r-2M)(M(r-4M) + (2r-5M)k^2r^3)}{(k^2r^3+2M)^2}\frac{dH_z}{dr} + \frac{r(r-2M)^2}{k^2r^3+2M}\frac{d^2H_z}{dr^2}. \end{aligned} \tag{3.17}$$

Finally, one can use the above expressions to obtain a decoupled ODE for $\mathfrak{h} := H_z$, namely,

$$\frac{d^2\mathfrak{h}}{dr^2}(r) + P_k(r)\frac{d\mathfrak{h}}{dr}(r) + \left(Q_k(r) - \frac{\mu^2 r^2}{(r-2M)^2}\right)\mathfrak{h}(r) = 0, \tag{3.18}$$

with

$$P_k(r) := \frac{12M}{r(k^2r^3+2M)} - \frac{5}{r} + \frac{1}{r-2M}, \tag{3.19}$$

$$Q_k(r) := \frac{6M}{r^2(r-2M)} - \frac{rk^2}{r-2M} - \frac{12M^2}{r^2(r-2M)(k^2r^3+2M)}. \tag{3.20}$$

Conversely, given any $C^2((2M,\infty))$ solution $\mathfrak{h}(r)$ to ODE (3.5) with $k \neq 0$ and $\mu \neq 0$, one can define $H_z(r) = \mathfrak{h}(r)$. Since $k \neq 0$, one can use Eqs. (3.15)–(3.17) to construct $H_t(r)$, $H_r(r)$, and $H_v(r)$. These then define the components of a mode solution h in spherical gauge (3.2). Explicitly,

$$h = e^{\mu t+ikz}\begin{pmatrix} H_t(r) & \mu H_v(r) & 0 & 0 & 0 \\ \mu H_v(r) & H_r(r) & 0 & 0 & -ikH_v(r) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -ikH_v(r) & 0 & 0 & H_z(r) \end{pmatrix}. \tag{3.21}$$

If ODE (3.5) is satisfied and (3.15)–(3.17) define H_r, H_v, and H_t, then Eqs. (3.8)–(3.14) are also satisfied. Therefore, a mode solution h constructed in this manner solves the linearized vacuum Einstein equation (2.4) on the exterior $\mathcal{E}_A$ of the Schwarzschild black string $\mathrm{Sch}_4 \times \mathbb{R}$. If $kR \in \mathbb{Z}$, then this construction also gives a mode solution h, which solves the linearized vacuum Einstein equation (2.4) on the exterior $\mathcal{E}_A$ of the Schwarzschild black string $\mathrm{Sch}_4 \times \mathbb{S}^1_R$.

□

Remark. If $k = 0$ and $\mu \neq 0$, then one can add an additional pure gauge solution h_{pg} to a mode solution h in spherical gauge (3.2) such that $h + h_{\mathrm{pg}}$ is also in spherical gauge (3.2) with $H_t(r) \equiv 0$. The relevant choice of pure gauge solution is given by $(h_{\mathrm{pg}})_{ab} = 2\nabla_{(a}\xi_{b)}$ with

$$\xi = e^{\mu t}\left(-\frac{H_t(r)}{2\mu}, 0, 0, 0, 0\right). \tag{3.22}$$

A mode solution h in spherical gauge with $H_t(r) \equiv 0$ satisfying the linearized vacuum Einstein equation (2.4) on the exterior $\mathcal{E}_A$ of the Schwarzschild black string is then again equivalent to the system of ODE (3.8)–(3.14) [with $k = 0$ and $H_t \equiv 0$] being satisfied. Equations (3.8) and (3.10) are automatically satisfied by $k = 0$. Eq. (3.12) automatically gives the decoupled equation (3.5) for H_z. Then, Eq. (3.9) can be solved for H_r in terms of H_z and $\frac{dH_z}{dr}$. This gives the relation in Eq. (3.15) for H_r with $k = 0$. Equation (3.11) can be used to solve for H_v in terms of H_z and $\frac{dH_z}{dr}$. At this point, the equations (3.13) and (3.14) are automatically satisfied. Therefore, again a solution to ODE (3.5) induces a mode solution in spherical gauge with $H_t = 0$.

C. Excluding pure gauge perturbations

This section contains two proofs that if $k \neq 0$ and $\mu \neq 0$, then a non-trivial mode solution h of the form (3.2) cannot be a pure gauge solution. One can prove this directly via the following proposition:

Proposition 3.3. Suppose $k \neq 0$ and $\mu \neq 0$. A non-trivial mode solution h in spherical gauge (3.2) of the linearized vacuum Einstein equation (2.4) on the exterior $\mathcal{E}_A$ of the Schwarzschild black string $\mathrm{Sch}_4 \times \mathbb{R}$ or $\mathrm{Sch}_4 \times \mathbb{S}^1_R$ cannot be pure gauge.

Proof. If h is pure gauge, it must be possible to write $h_{ab} = 2\nabla_{(a}\xi_{b)}$ for some vector field ξ. Therefore, one finds

$$h_{zz} = H_z(r)e^{\mu t+ikz} \implies 2\partial_z\xi_z = H_z(r)e^{\mu t+ikz}, \tag{3.23}$$

$$h_{z\theta} = 0 \implies \partial_\theta\xi_z + \partial_z\xi_\theta = 0. \tag{3.24}$$

Applying ∂_z to Eq. (3.24), using that partial derivatives commute and that, from Eq. (3.23), $\partial_z\xi_z$ clearly does not depend on θ gives

$$\partial_z^2\xi_\theta = 0. \tag{3.25}$$

Next, $h_{\theta\theta} = 0$ implies

$$\partial_\theta\xi_\theta - \Gamma^r_{\theta\theta}\xi_r = 0. \tag{3.26}$$

From Appendix A, $\Gamma^r_{\theta\theta} = (r-2M)$. Hence, taking two derivatives of (3.26) in the z direction and using $\partial_z^2\xi_\theta = 0$ give

$$\Gamma^r_{\theta\theta}\partial_z^2\xi_r = (r-2M)\partial_z^2\xi_r = 0. \tag{3.27}$$

Therefore, $\partial_z^2\xi_r = 0$ on $\mathcal{E}_A$.

From the h_{rr} component, one has,

$$2\partial_r\xi_r - 2\Gamma^r_{rr}\xi_r = 2\partial_r\xi_r + \frac{2M}{r(r-2M)}\xi_r = H_r e^{\mu t+ikz}, \tag{3.28}$$

where one uses $\Gamma^r_{rr} = -\frac{M}{r(r-2M)}$ from Appendix A. Taking the second z derivative of Eq. (3.28) and using $\partial_z^2\xi_r = 0$ on $\mathcal{E}_A$ give

$$k^2 H_r = 0 \quad \text{on} \quad \mathcal{E}_A. \tag{3.29}$$

Since $k \neq 0$, this implies $H_r \equiv 0$ on the exterior $\mathcal{E}_A$. Since $k \neq 0$ and $\mu \neq 0$, Eq. (3.8) implies that if $H_r = 0$ on $\mathcal{E}_A$, then $H_v \equiv 0$ on $\mathcal{E}_A$. Using the h_{zr} component, one finds

$$\partial_z\xi_r + \partial_r\xi_z = -ikH_v e^{\mu t+ikz} = 0 \implies \partial_r(\partial_z\xi_z) = 0 \implies \frac{dH_z}{dr} = 0 \quad \text{on} \quad \mathcal{E}_A, \tag{3.30}$$

where one uses the identity $\partial_z^2\xi_r = 0$ on $\mathcal{E}_A$ in the first implication and that $\partial_z\xi_z = H_z(r)e^{\mu t+ikz}$ in the second implication. The linearized vacuum Einstein equation (2.4) under this ansatz [Eq. (3.9)] then implies $H_z \equiv 0$ on $\mathcal{E}_A$, and therefore, from equations (3.10) and (3.11), $H_t \equiv 0$ on $\mathcal{E}_A$. Hence, $h \equiv 0$ on $\mathcal{E}_A$.

□

Perhaps more satisfactorily, one can establish that if h is a non-trival mode solution in spherical gauge (3.2) with $k \neq 0$ and $\mu \neq 0$, then the *tztz*-component of the linearized Weyl tensor $\overset{(1)}{W}$ is non-vanishing. Moreover, if h has $\mu > 0$, then $\overset{(1)}{W}_{tztz}$ grows exponentially. Since $\overset{(1)}{W}_{tztz}$ is gauge invariant, this behavior persists in all gauges. More precisely, one has the following proposition:

Proposition 3.4. Suppose $k \neq 0$, $\mu \neq 0$, and h is a non-trivial mode solution in spherical gauge (3.2) of the linearized vacuum Einstein equation (2.4) on the exterior $\mathcal{E}_A$ of the Schwarzschild black string $\mathrm{Sch}_4 \times \mathbb{R}$ *or* $\mathrm{Sch}_4 \times \mathbb{S}^1_R$. *Then, $\overset{(1)}{W}_{tztz}$ is non-vanishing and h is not pure gauge. Moreover, if $\mu > 0$ then $\overset{(1)}{W}_{tztz}$ also grows exponentially.*

Proof. By Proposition 2.3, $\overset{(1)}{W}_{tztz}$ is gauge invariant. Hence, if $\overset{(1)}{W}_{tztz}$ is non-zero when evaluated on a non-trivial mode solution h in spherical gauge (3.2), h cannot be pure gauge. Using Proposition 2.2 gives the following expression for $\overset{(1)}{W}_{tztz}$:

$$\overset{(1)}{W}_{tztz} = \frac{1}{2}e^{\mu t+ikz}\left(k^2 H_t(r) - \frac{2Mk^2(r-2M)}{r^3}H_v(r) + \frac{M(r-2M)}{r^3}\frac{dH_z}{dr}(r) - \mu^2 H_z(r)\right). \tag{3.31}$$

If $k \neq 0$ and $\mu \neq 0$, one can use Eqs. (3.15)-(3.17) and ODE (3.5) to simplify this to

$$\begin{aligned}\overset{(1)}{W}_{tztz} = -\,e^{\mu t+ikz}\Bigg(&\frac{M(r-2M)(k^2r^3(3r-7M)-2M^2)}{r^3(k^2r^3+2M)^2}\frac{dH_z}{dr}(r)\\ &+\frac{M(k^4r^3(r-2M)+k^2(\mu^2r^4-Mr+2M^2)+2M\mu^2 r)}{r(k^2r^3+2M)^2}H_z(r)\Bigg).\end{aligned} \tag{3.32}$$

Suppose $\overset{(1)}{W}_{tztz} \equiv 0$ identically, then

$$\frac{dH_z}{dr}(r) = \frac{r^2(Mr(2k^4r^2+k^2-2\mu^2)-k^2(\mu^2+k^2)r^4-2M^2k^2)}{(r-2M)(3k^2r^4-7Mk^2r^3-2M^2)}H_z(r). \tag{3.33}$$

Substituting this into ODE (3.5) gives that either

$$k^4r^3(r-2M)^2 + Mr(4r-9M)\mu^2 + r^5\mu^4 + k^2(r-2M)(2r^4\mu^2 - 2Mr + 5M^2) = 0 \tag{3.34}$$

for all $r \in (2M, \infty)$ or $H_z(r) \equiv 0$. If $\mu \neq 0$ and $k \neq 0$, then the polynomial in Eq. (3.34) has at most five roots in $r \in (2M, \infty)$. Therefore, if $\overset{(1)}{W}_{tztz} = 0$, then $H_z(r) = 0$, which is a contradiction. Moreover, since $\overset{(1)}{W}_{tztz} \neq 0$, it is clear from Eq. (3.32) that if $\mu > 0$, then $\overset{(1)}{W}_{tztz}$ grows exponentially.

□

D. Admissible boundary conditions

One can construct two sets of distinguished solutions to ODE (3.5) associated with the "end points" of the interval $(2M, \infty)$. Note that by Definition B.1 from Appendix B, $r = 2M$ is a regular singularity, as $2M$ is not an ordinary point and

$$(r-2M)P_k(r) \quad \text{and} \quad (r-2M)^2\left(Q_k(r) - \frac{\mu^2 r^2}{(r-2M)^2}\right) \tag{3.35}$$

are analytic near $r = 2M$. By Definition B.3, ODE (3.5) has an irregular singularity at infinity since there exist convergent series expansions

$$P_k(r) = \sum_{n=0}^{\infty} \frac{p_n}{z^n} \quad \text{and} \quad Q_k(r) - \frac{\mu^2 r^2}{(r-2M)^2} = \sum_{n=0}^{\infty} \frac{q_n}{z^n} \tag{3.36}$$

in a neighborhood of infinity with $p_0 = 0$, $p_1 = -4$, $q_0 = -(k^2 + \mu^2)$, and $q_1 = -2M(k^2 + 2\mu^2)$. The asymptotic analysis of the ODEs around these points is examined in Secs. III D 1 and III D 2. This analysis of ODE (3.5) near $r = 2M$ and $r = \infty$ will lead to the identification of the admissible boundary conditions for a mode solution h in spherical gauge (3.2) of the linearized Einstein vacuum equation (2.4).

1. The future event horizon $\mathcal{H}_A^+$

The goal of this section is to identify the admissible boundary conditions for a solution $\mathfrak{h}$ to ODE (3.5) near $r = 2M$. This requires one to understand the behavior near $r = 2M$ of the mode solution h in spherical gauge (3.2) of the linearized vacuum Einstein equation (2.4), which results (through the construction in Proposition 3.2) from $\mathfrak{h}$.

Associated with the future event horizon $\mathcal{H}_A^+$, there exists a basis $\mathfrak{h}^{2M,\pm}$ for solutions to ODE (3.5). From $\mathfrak{h}^{2M,\pm}$, one can examine the behavior near $r = 2M$ of any mode solution h in spherical gauge (3.2) with $\mu \neq 0$ and $k \neq 0$ through Proposition 3.2. A mode solution h in spherical gauge (3.2) with $\mu > 0$ and $k \neq 0$ constructed from $\mathfrak{h}^{2M,-}$ never smoothly extends to the future event horizon. A mode solution h in spherical gauge (3.2) with $\mu > 0$ and $k \neq 0$ constructed from $\mathfrak{h}^{2M,+}$ also does not smoothly extend to the future event horizon unless μ satisfies particular conditions. However, if h is a mode solution in spherical gauge (3.2) with $\mu > 0$ and $k \neq 0$ constructed from $\mathfrak{h}^{2M,+}$, then after the addition of a pure gauge solution h_{pg}, it turns that out one can smoothly extend $h + h_{\mathrm{pg}}$ to the future event horizon. Moreover, it will be shown that $h + h_{\mathrm{pg}}$ satisfies the harmonic/transverse-traceless gauge (1.10) conditions. This will be the content of Proposition 3.5.

First, some preliminaries: The coefficients of ODE (3.5) extend meromorphically to $r = 2M$ and behave asymptotically as

$$P_k(r) = \frac{1}{r-2M} + \mathcal{O}(1) \qquad Q_k(r) - \frac{\mu^2 r^2}{(r-2M)^2} = -\frac{4M^2\mu^2}{(r-2M)^2} + \mathcal{O}\left(\frac{1}{r-2M}\right). \tag{3.37}$$

Hence, one may write ODE (3.5) as

$$\frac{d^2\mathfrak{h}}{dr^2} + \left(\frac{1}{r-2M} + \mathcal{O}(1)\right)\frac{d\mathfrak{h}}{dr} - \left(\frac{4M^2\mu^2}{(r-2M)^2} + \mathcal{O}\left(\frac{1}{r-2M}\right)\right)\mathfrak{h} = 0. \tag{3.38}$$

From Appendix B, the indicial equation associated with the ODE (3.38) is

$$I(\alpha) = \alpha^2 - 4M^2\mu^2, \tag{3.39}$$

which has roots

$$\alpha_\pm := \pm 2M\mu. \tag{3.40}$$

If $\alpha_+ - \alpha_- = 4M\mu \notin \mathbb{Z}$, then one can deduce from Theorem B.1 the asymptotic basis for solutions near $r = 2M$. If $\alpha_+ - \alpha_- = 4M\mu \in \mathbb{Z}$, then the relevant result for the asymptotic basis of solutions is Theorem B.2. Combining the results of Theorems B.1 and B.2, one has the following basis for solutions for $\mu > 0$:

$$\mathfrak{h}^{2M,+}(r) := (r-2M)^{2M\mu} \sum_{n=0}^{\infty} a_n^+ (r-2M)^n, \tag{3.41}$$

$$\mathfrak{h}^{2M,-}(r) := \begin{cases} \sum_{n=0}^{\infty} a_n^- (r-2M)^{n-2M\mu} + C_N \mathfrak{h}^{2M,+}(r)\ln(r-2M) & \text{if} \quad 4M\mu = N \in \mathbb{Z}_{>0} \\ (r-2M)^{-2M\mu} \sum_{n=0}^{\infty} a_n^- (r-2M)^n & \text{otherwise,} \end{cases} \tag{3.42}$$

where the coefficients a_n^+, a_n^-, and the anomalous term C_N can be calculated recursively (see Theorems B.1 and B.2). A general solution to ODE (3.5) will be of the form

$$\mathfrak{h}(r) = k_1 \mathfrak{h}^{2M,+}(r) + k_2 \mathfrak{h}^{2M,-}(r), \tag{3.43}$$

with $k_1, k_2 \in \mathbb{R}$.

If $4M\mu$ is *not* an integer or $4M\mu$ is an integer and $C_N = 0$, then the asymptotic basis for solutions for $\mu > 0$ reduces to

$$\mathfrak{h}^{2M,+}(r) = (r-2M)^{2M\mu}\sum_{n=0}^{\infty} a_n^+(r-2M)^n, \tag{3.44}$$

$$\mathfrak{h}^{2M,-}(r) = (r-2M)^{-2M\mu}\sum_{n=0}^{\infty} a_n^-(r-2M)^n. \tag{3.45}$$

In Eqs. (3.44) and (3.45), the first order coefficients of the basis can be calculated to be

$$a_1^{\pm} = \frac{\pm\mu(20M^2k^2-1)+4M(\mu^2-k^2+4M^2\mu^2k^2+2M^2k^4)}{(1\pm 4M\mu)(4M^2k^2+1)}. \tag{3.46}$$

The main result of this section is the following:

Proposition 3.5. Suppose $\mu > 0, k \neq 0$, and let $\mathfrak{h}$ be a solution to ODE (3.5). Let h be the mode solution on the exterior $\mathcal{E}_A$ of the Schwarzschild black string $\mathrm{Sch}_4 \times \mathbb{R}$ constructed from $H_z = \mathfrak{h}$ in Proposition 3.2. Then, there exists a pure gauge solution h_{pg} such that $h + h_{\mathrm{pg}}$ extends to a smooth solution of the linearized vacuum Einstein equation (2.4) at the future event horizon $\mathcal{H}_A^+$ if $k_2 = 0$, where k_2 is defined in Eq. (3.43). Moreover, $h + h_{\mathrm{pg}}$ can be chosen to satisfy the harmonic/transverse-traceless gauge (1.10) conditions.

Remark. To determine admissible boundary conditions of $\mathfrak{h}$ at $r = 2M$, it is essential that one works in coordinates that extend regularly across this hypersurface. A good choice is ingoing Eddington–Finkelstein coordinates $(v, r, \theta, \varphi, z)$ defined by

$$v = t + r_*(r), \qquad r_*(r) = r + 2M\log|r-2M|. \tag{3.47}$$

Also note that for the boundary conditions to be admissible, one needs to consider all components of the mode solution h constructed from $\mathfrak{h}$ via Proposition 3.2. These remarks will be implemented in the Proof of Proposition 3.5.

Before proving the statement of Proposition 3.5, it is useful to prove the following lemma:

Lemma 3.6. Let h be a mode solution of the linearized vacuum Einstein equation (2.4) of the form

$$h_{\alpha\beta} = e^{\mu t+ikz}\begin{pmatrix} H_{tt}(r) & H_{tr}(r) & 0 & 0 & 0\\ H_{tr}(r) & H_{rr}(r) & 0 & 0 & 0\\ 0 & 0 & H_{\theta\theta}(r) & 0 & 0\\ 0 & 0 & 0 & H_{\theta\theta}(r)\sin^2\theta & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}. \tag{3.48}$$

Then, h satisfies the following harmonic/transverse-traceless gauge conditions:

$$\begin{cases} \nabla^a h_{ab} = 0,\\ g^{ab}h_{ab} = 0\end{cases} \tag{3.49}$$

if $k \neq 0$.

Proof. First, it is instructive to write out explicit expressions for $\nabla_c h_{ab}$ and $\nabla_c\nabla_d h_{ab}$ in coordinates. These are the following:

$$\nabla_\gamma h_{\alpha\beta} = \partial_\gamma h_{\alpha\beta} - \Gamma^\lambda_{\gamma\alpha}h_{\lambda\beta} - \Gamma^\lambda_{\gamma\beta}h_{\alpha\lambda}, \tag{3.50}$$

$$\begin{aligned}\nabla_\gamma\nabla_\delta h_{\alpha\beta} = {}& \partial_\gamma(\partial_\delta h_{\alpha\beta} - \Gamma^\lambda_{\delta\alpha}h_{\lambda\beta} - \Gamma^\lambda_{\delta\beta}h_{\alpha\lambda}) - \Gamma^\mu_{\gamma\delta}(\partial_\mu h_{\alpha\beta} - \Gamma^\lambda_{\mu\alpha}h_{\lambda\beta} - \Gamma^\lambda_{\mu\beta}h_{\alpha\lambda}),\\ & - \Gamma^\mu_{\gamma\alpha}(\partial_\delta h_{\mu\beta} - \Gamma^\lambda_{\delta\mu}h_{\lambda\beta} - \Gamma^\lambda_{\delta\beta}h_{\mu\lambda}) - \Gamma^\mu_{\gamma\beta}(\partial_\delta h_{\alpha\mu} - \Gamma^\lambda_{\delta\alpha}h_{\lambda\mu} - \Gamma^\lambda_{\delta\mu}h_{\alpha\lambda}).\end{aligned} \tag{3.51}$$

If one takes ansatz (3.48) and $\alpha = z$ in Eq. (3.51), then since $h_{z\beta} = 0$ for all $\beta \in \{t, r, \theta, \varphi, z\}$ and, from Appendix A, $\Gamma^\lambda_{z\beta} = 0$ for all $\beta, \lambda \in \{t, r, \theta, \varphi, z\}$,

$$\nabla_\gamma\nabla_\delta h_{\alpha\beta} = 0 \qquad (\alpha = z). \tag{3.52}$$

Hence,

$$g^{\gamma\delta}\nabla_\gamma\nabla_\delta h_{\alpha\beta} = 0 \qquad (\alpha = z), \tag{3.53}$$

$$g^{\delta\beta}\nabla_\gamma\nabla_\delta h_{\alpha\beta} = 0 \qquad (\alpha = z). \tag{3.54}$$

Consider the linearized vacuum Einstein equation (2.4) in coordinates

$$g^{\gamma\delta}\nabla_\gamma\nabla_\delta h_{\alpha\beta} + \nabla_\alpha\nabla_\beta h - \nabla_\alpha\nabla^\gamma h_{\beta\gamma} - \nabla_\beta\nabla^\gamma h_{\alpha\gamma} + 2R_\alpha{}^\gamma{}_\beta{}^\delta h_{\gamma\delta} = 0. \tag{3.55}$$

Since from Eqs. (3.53) and (3.54) and from Appendix A, $R_{z\beta\gamma\delta} = 0$, it follows that the linearized vacuum Einstein equation in local coordinates with $\alpha = z$ and under ansatz (3.48) reduces to

$$\nabla_z(\nabla_\beta h - \nabla^\gamma h_{\beta\gamma}) = 0. \tag{3.56}$$

Furthermore, $\nabla_z = \partial_z$, so using the explicit z-dependence of ansatz (3.48), Eq. (3.56) reduces to

$$k(\nabla_\beta h - \nabla^\gamma h_{\beta\gamma}) = 0. \tag{3.57}$$

Since $k \neq 0$, the harmonic gauge condition

$$\nabla_\beta h - \nabla^\gamma h_{\beta\gamma} = 0 \tag{3.58}$$

is satisfied. If $\beta = z$, then using Eq. (3.50) and $\nabla_z = \partial_z$, Eq. (3.58) reduces to

$$\partial_z h = kh = 0 \implies h = 0 \tag{3.59}$$

since $k \neq 0$. Substituting (3.59) into Eq. (3.58) gives the transverse condition

$$\nabla^\gamma h_{\beta\gamma} = 0. \tag{3.60}$$

□

Proof of Proposition 3.5. Consider $H_z^{2M,\pm} := \mathfrak{h}^{2M,\pm}$ where $\mathfrak{h}^{2M,\pm}$ are given by Eqs. (3.44) and (3.45) with first order coefficients (3.46). Taking $k_2 = 0$ is equivalent to examining the basis element $H_z^{2M,+}$. Since $\mu > 0$ and $k \neq 0$, one can use Proposition 3.2 to construct the components H_t, H_r, and H_v associated to $H_z^{2M,\pm}$. Substituting the basis into Eqs. (3.15)–(3.17), one finds

$$H_r^{2M,\pm} = (r-2M)^{-2\pm 2M\mu}\left(\frac{M^2(\pm 4M\mu - 1)}{1+4M^2k^2} + \frac{M(4M^2(2\mu^2+k^2) \pm 6M\mu - 1)}{2(1+4M^2k^2)}(r-2M) + \mathcal{O}((r-2M)^2)\right), \tag{3.61}$$

$$H_t^{2M,\pm} = (r-2M)^{\pm 2M\mu}\left(\frac{(1+4M\mu)(4M\mu-1)}{4(1+4M^2k^2)} + \frac{3+4M^2(8\mu^2-k^2) \pm 2M\mu(8M^2(2\mu^2+k^2)-11)}{8M(1+4M^2k^2)}(r-2M) + \mathcal{O}((r-2M)^2)\right), \tag{3.62}$$

$$H_v^{2M,\pm} = (r-2M)^{-1+2M\mu}\left(\frac{M^2(\pm 4M\mu - 1)}{1+4M^2k^2} + \frac{M(2M^2(2\mu^2+k^2) - 1 \pm 5M\mu)}{1+4M^2k^2}(r-2M) + \mathcal{O}((r-2M)^2)\right). \tag{3.63}$$

Consider a pure gauge solution $h_{\mathrm{pg}} = 2\nabla_{(a}\xi_{b)}$ generated by the following vector field

$$\xi = e^{\mu t + ikz}\left(-\frac{\mu H_z(r)}{2k^2}, \frac{2k^2 H_v(r) - \frac{dH_z}{dr}(r)}{2k^2}, 0, 0, \frac{iH_z(r)}{2k}\right), \tag{3.64}$$

where H_v is defined via Eq. (3.16). This gives a new solution to the linearized vacuum Einstein equation (2.4),

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} + 2\nabla_{(\mu}\xi_{\nu)} = e^{\mu t + ikz}\begin{pmatrix} \tilde{H}_{tt}(r) & \tilde{H}_{tr}(r) & 0 & 0 & 0 \\ \tilde{H}_{tr}(r) & \tilde{H}_{rr}(r) & 0 & 0 & 0 \\ 0 & 0 & \tilde{H}_{\theta\theta}(r) & 0 & 0 \\ 0 & 0 & 0 & \tilde{H}_{\theta\theta}(r)\sin^2\theta & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.65}$$

with the following expressions for the matrix components:

$$\tilde{H}_{tt}(r) = c_1(r)H_z(r) + c_2(r)\frac{dH_z}{dr}(r), \tag{3.66}$$

$$\tilde{H}_{\theta\theta}(r) = c_3(r)H_z(r) + c_4(r)\frac{dH_z}{dr}(r), \tag{3.67}$$

$$\tilde{H}_{rr}(r) = \frac{r^2}{(r-2M)^2}\tilde{H}_{tt}(r) - \frac{2}{r(r-2M)}\tilde{H}_{\theta\theta}(r), \tag{3.68}$$

$$\tilde{H}_{tr}(r) = -\frac{2M\mu}{k^2(2M+r^3k^2)}\left(\frac{dH_z}{dr}(r) - \frac{M}{r(r-2M)}H_z(r)\right), \tag{3.69}$$

where

$$\begin{aligned} c_1(r) &:= \frac{6M^2(r-2M)}{r(k^2r^3+2M)^2} - \frac{2M(r-2M)}{r(k^2r^3+2M)} + \frac{\mu^2 r^3}{k^2r^3+2M} - \frac{\mu^2}{k^2}, \\ c_2(r) &:= \frac{M(r-2M)}{k^2r^3} - \frac{M(r-2M)}{k^2r^3+2M} - \frac{6M(4M^2-4Mr+r^2)}{(k^2r^3+2M)^2}, \\ c_3(r) &:= -\frac{Mr^2}{k^2r^3+2M}, \qquad c_4(r) := \frac{r^3(r-2M)}{k^2r^3+2M} - \frac{r-2M}{k^2}. \end{aligned} \tag{3.70}$$

Note that Eqs. (3.5) and (3.15)–(3.17) have been used to derive Eqs. (3.66)–(3.69). By Lemma 3.6, this new mode solution (3.65) satisfies the harmonic/transverse-traceless gauge,

$$\begin{cases} g^{\mu\nu}\tilde{h}_{\mu\nu} = 0, \\ \nabla^{\mu}\tilde{h}_{\mu\nu} = 0. \end{cases} \tag{3.71}$$

As remarked above, to determine admissible boundary conditions of $\mathfrak{h}$ at $r = 2M$, it is essential that one works in coordinates that extend regularly across this hypersurface. Moreover, to identify the boundary conditions to be admissible, one needs to consider all components of the mode solution h constructed from $\mathfrak{h}$ via Proposition 3.2. The following formulas give the transformation to ingoing Eddington–Finkelstein coordinates for the components of the mode solution h defined in Eq. (3.65):

$$\begin{aligned} \tilde{H}'_{vv} &= \left(\frac{\partial t}{\partial v}\right)^2 \tilde{H}_{tt}, \\ \tilde{H}'_{vr} &= \left(\frac{\partial t}{\partial v}\right)\left(\frac{\partial r}{\partial r}\right)\tilde{H}_{tr} + \left(\frac{\partial t}{\partial v}\right)\left(\frac{\partial t}{\partial r}\right)\tilde{H}_{tt} \\ &= \tilde{H}_{tr} - \frac{r}{r-2M}\tilde{H}_{tt}, \\ \tilde{H}'_{rr} &= \left(\frac{\partial t}{\partial r}\right)^2 \tilde{H}_{tt} + \left(\frac{\partial t}{\partial r}\right)\left(\frac{\partial r}{\partial r}\right)\tilde{H}_{tr} + \left(\frac{\partial r}{\partial r}\right)^2 \tilde{H}_{rr} \\ &= \frac{r^2}{(r-2M)^2}\tilde{H}_{tt} - \frac{r}{r-2M}\tilde{H}_{tr} + \tilde{H}_{rr}, \end{aligned} \tag{3.72}$$

where one uses $t = v - r_*(r)$ with $r_*(r) = r + 2M\log|r-2M|$. Explicitly, Eq. (3.72) can be computed to be

$$\begin{aligned} \tilde{H}'_{vv} = {} & \frac{2M(2M\mu^2 r + k^2(\mu^2 r^4 - Mr + 2M^2) + k^4r^3(r-2M))}{r(k^3r^3+2Mk)^2}H_z(r) \\ & - \frac{2M(r-2M)(k^2r^3(3r-7M)-2M^2)}{r^3(k^3r^3+2Mk)^2}\frac{dH_z}{dr}(r), \end{aligned} \tag{3.73}$$

$$\begin{aligned} \tilde{H}'_{vr} = {} & \left(\frac{\mu(\mu r^2+M)}{rk^2(r-2M)} - \frac{\mu^2r^4 + M\mu r^2 - 2M(r-2M)}{(r-2M)(k^2r^3+2M)} - \frac{6M^2}{(k^2r^3+2M)^2}\right)H_z \\ & + \left(\frac{6Mr(r-2M)}{(k^2r^3+2M)^2} + \frac{r(\mu r^2+M)}{k^2r^3+2M} - \frac{\mu r^2+M}{k^2r^2}\right)\frac{dH_z}{dr}, \end{aligned} \tag{3.74}$$

$$
\begin{aligned}
\tilde{H}'_{rr} = &\left(\frac{2r(M\mu r^2+\mu^2r^4-M(r-2M))}{(r-2M)^2(k^2r^3+2M)}+\frac{12M^2r}{(k^2r^3+2M)^2(r-2M)}-\frac{2\mu(\mu r+M)}{k^2(r-2M)^2}\right)H_z \\
&+\left(\frac{2(\mu r^2+r-M)}{k^2r(r-2M)}-\frac{12Mr^2}{(k^2r^3+2M)^2}-\frac{2r^2(\mu r^2+r-M)}{(r-2M)(k^2r^3+2M)}\right)H'_z,
\end{aligned} \tag{3.75}
$$

$$
\tilde{H}'_{\theta\theta} = -\frac{Mr^2}{k^2r^3+2M}H_z(r)-\frac{2M(r-2M)}{k^4r^3+2Mk^2}\frac{dH_z}{dr}(r), \tag{3.76}
$$

where ODE (3.5) with $\mathfrak{h}=H_z$ has been used. To determine the behavior of these new metric perturbation components close to the future event horizon $\mathcal{H}_A^+$, one must substitute $H_z^{2M,\pm}(r) := \mathfrak{h}^{2M,\pm}(r)$ from Eqs. (3.41) and (3.42). Substituting $H_z^{2M,\pm}(r) := \mathfrak{h}^{2M,\pm}(r)$ from Eqs. (3.44) and (3.45) into these expressions gives leading order behavior close to the future event horizon $\mathcal{H}_A^+$ determined by the relations

$$
\tilde{H}_{vv}^{2M,\pm} = f_{vv}(r)(r-2M)^{\pm 2M\mu}, \tag{3.77}
$$

$$
\tilde{H}_{vr}^{2M,\pm} = \left(\frac{(\mu\mp\mu)(1+4M\mu)}{2k^2(1+4M^2k^2)}(r-2M)^{-1}+f_{vr}(r)\right)(r-2M)^{\pm 2M\mu}, \tag{3.78}
$$

$$
\tilde{H}_{rr}^{2M,\pm} = \left(\frac{-2(1\mp 1)M\mu(1+4M\mu)}{k^2(1+4M^2k^2)}(r-2M)^{-2}+k_\pm(r-2M)^{-1}+f_{rr}(r)\right)(r-2M)^{\pm 2M\mu}, \tag{3.79}
$$

$$
\tilde{H}_{\theta\theta}^{2M,\pm} = f_{\theta\theta}(r)(r-2M)^{\pm 2M\mu}, \tag{3.80}
$$

with f_{vv}, f_{vr}, f_{rr}, $f_{\theta\theta}$ being smooth functions of $r\in[2M,\infty)$, which are non-vanishing at $2M$, $k_+=0$, and k_- being a non-zero constant depending on k, M and μ. Therefore, multiplying $\tilde{H}_{vv}^{2M,+}$, $\tilde{H}_{vr}^{2M,+}$, $\tilde{H}_{rr}^{2M,+}$, and $\tilde{H}_{\theta\theta}^{2M,+}$ by $e^{\mu t}=e^{\mu v}e^{-\mu r}(r-2M)^{-2M\mu}$ gives

$$
e^{\mu t+ikz}\tilde{H}_{vv}^{2M,+} = f_{vv}(r)e^{\mu v-\mu r+ikz}, \tag{3.81}
$$

$$
e^{\mu t+ikz}\tilde{H}_{vr}^{2M,+} = f_{vr}(r)e^{\mu v-\mu r+ikz}, \tag{3.82}
$$

$$
e^{\mu t+ikz}\tilde{H}_{rr}^{2M,+} = f_{rr}(r)e^{\mu v-\mu r+ikz}, \tag{3.83}
$$

$$
e^{\mu t+ikz}\tilde{H}_{\theta\theta}^{2M,+} = f_{\theta\theta}(r)e^{\mu v-\mu r+ikz}, \tag{3.84}
$$

which can indeed be smoothly extended to the future event horizon $\mathcal{H}_A^+$.

□

Remark. The form of the pure gauge solution defined by Eq. (3.64) can be derived as follows: From Lemma 3.6, a mode solution $\tilde{h}$ of the form (3.48) satisfies the harmonic/transverse-traceless (1.10) gauge conditions. Take a mode solution h in spherical gauge (3.2) add the pure gauge solution $h_{\text{pg}}=2\nabla_{(a}\xi_{b)}$ for some vector field

$$
\xi = e^{\mu t+ikz}\zeta, \tag{3.85}
$$

where ζ is a vector field which depends only on r. From a direct calculation of $h+h_{\text{pg}}$, one can see that to obtain a solution $\tilde{h}$ of the form (3.48), ζ must be given by Eq. (3.64).

Remark. To explicitly see the singular behavior of the mode solution $h^\pm$ in spherical gauge (3.2) with $\mu>0$ and $k\neq 0$ associated, via Proposition 3.2, to either $\mathfrak{h}^{2M,\pm}$, consider directly transforming to ingoing Eddington–Finkelstein coordinates. This transformation gives the following basis elements:

$$
H_{rr}^{2M,\pm\prime} = \left(\frac{\partial t}{\partial r}\right)^2 H_t^{2M,\pm}+2\left(\frac{\partial t}{\partial r}\right)\mu H_v^{2M,\pm}+H_r^{2M,\pm}(r), \tag{3.86}
$$

$$
H_{vv}^{2M,\pm\prime} = H_t^{2M,\pm}(r), \tag{3.87}
$$

$$
H_{vr}^{2M,\pm\prime} = \left(\frac{\partial t}{\partial r}\right)H_t^{2M,\pm}(r)+\mu H_v^{2M,\pm}(r), \tag{3.88}
$$

$$
H_{zz}^{2M,\pm\prime} = H_z^{2M,\pm}(r), \tag{3.89}
$$

where $H_v^{2M,\pm}$, $H_t^{2M,\pm}$, and $H_r^{2M,\pm}$ are the basis for solutions for H_v, H_t, and H_r constructed from Proposition (3.2). These relevant expressions can be found from Eqs. (3.15)–(3.17).

First, if $4M\mu$ is a positive integer and the coefficient C_N does not vanish, then by Eq. (3.89), the basis element $H_{zz}^{2M,-\prime}(r) = H_z^{2M,-} = \mathfrak{h}^{2M,-}$ has an essential logarithmic divergence and is therefore always singular at the future event horizon $\mathcal{H}_A^+$.

If $C_N = 0$ or $4M\mu$ is not a positive integer, then the basis elements $H_z^{2M,\pm} = \mathfrak{h}^{2M,\pm}$ are given by Eqs. (3.44) and (3.45) with first order coefficients (3.46). Substituting the basis into Eqs. (3.15)–(3.17) for the other metric perturbation component, one finds

$$H_r^{2M,\pm} = (r-2M)^{-2\pm 2M\mu}\left(\frac{M^2(\pm 4M\mu - 1)}{1+4M^2k^2} + \frac{M(4M^2(2\mu^2+k^2) \pm 6M\mu - 1)}{2(1+4M^2k^2)}(r-2M) + \mathcal{O}((r-2M)^2)\right), \tag{3.90}$$

$$H_t^{2M,\pm} = (r-2M)^{\pm 2M\mu}\left(\frac{(1+4M\mu)(4M\mu-1)}{4(1+4M^2k^2)} + \frac{3+4M^2(8\mu^2-k^2) \pm 2M\mu(8M^2(2\mu^2+k^2)-11)}{8M(1+4M^2k^2)}(r-2M) + \mathcal{O}((r-2M)^2)\right), \tag{3.91}$$

$$H_v^{2M,\pm} = (r-2M)^{-1+2M\mu}\left(\frac{M^2(\pm 4M\mu - 1)}{1+4M^2k^2} + \frac{M(2M^2(2\mu^2+k^2) - 1 \pm 5M\mu)}{1+4M^2k^2}(r-2M) + \mathcal{O}((r-2M)^2)\right) \tag{3.92}$$

Transforming to ingoing Eddington–Finkelstein coordinates gives

$$\begin{aligned} H_{rr}^{2M,\pm\prime} = {} & (r-2M)^{-2\pm 2M\mu}\Bigg(\frac{2M^2(1-2M\mu(1\mp 1))(4M\mu-1)}{1+4M^2k^2} \\ & + \frac{2M^2\mu\big((3\mp 4) + 2(9\mp 7)M\mu - (1\mp 1)4M^2(2\mu^2+k^2)\big)}{1+4M^2k^2}(r-2M) \\ & + \mathcal{O}((r-2M)^2)\Bigg), \end{aligned} \tag{3.93}$$

$$H_{vv}^{2M,\pm\prime} = (r-2M)^{\pm 2M\mu}\left(\frac{(1+4M\mu)(4M\mu-1)}{4(1+4M^2k^2)} + \mathcal{O}(r-2M)\right), \tag{3.94}$$

$$H_{vr}^{2M,\pm\prime} = (r-2M)^{-1\pm 2M\mu}\left(\frac{M(2M\mu(1\mp 2)-1)(\pm 4M\mu - 1)}{2(1+4M^2k^2)} + \mathcal{O}(r-2M)\right), \tag{3.95}$$

$$H_{zz}^{2M,\pm\prime} = (r-2M)^{\pm 2M\mu}(1+\mathcal{O}(r-2M)). \tag{3.96}$$

Note that the full mode solution h constructed from Proposition 3.2 involves a factor of $e^{\mu t} = e^{\mu v}e^{-\mu r}(r-2M)^{-2M\mu}$, so after multiplication by this exponential factor, one can see that the basis elements $H_{\mu\nu}^{2M,-\prime}$ are always singular, i.e., a solution with $k_2 \neq 0$ is always singular at the future event horizon. The components $e^{\mu t}H_{vv}^{2M,+\prime}$ and $e^{\mu t}H_z^{2M,+\prime}$ are unconditionally smooth. However, in general, the components $e^{\mu t}H_{rr}^{2M,+\prime}$ and $e^{\mu t}H_{vr}^{2M,+\prime}$ remain singular at the future event horizon $\mathcal{H}_A^+$ unless $4M\mu = 1$ or $-2+2M\mu \in \mathbb{N}\cup\{0\}$ or $-2+2M\mu > 2$. [In Appendix E, it is shown that for existence of a solution $\mathfrak{h}$ with $\mu > 0$, which has $k_2 = 0$ and is finite at infinity (see Sec. III D 2), then $\mu < \frac{3}{16M}\sqrt{\frac{3}{2}} < \frac{1}{4M}$.] Hence, neither basis perturbation $h^\pm$ in spherical gauge (3.2) extends, in general, smoothly across the future event horizon $\mathcal{H}_A^+$.

2. Spacelike infinity i_A^0

The goal of this section is to identify the admissible boundary conditions for a solution $\mathfrak{h}$ to ODE (3.5) as $r \to \infty$. This requires one to understand the behavior as $r \to \infty$ of the mode solution h in spherical gauge (3.2) of the linearized vacuum Einstein equation (2.4), which results (through the construction in Proposition 3.2) from $\mathfrak{h}$.

In this section, a basis for solution $\mathfrak{h}^{\infty,\pm}$ associated with $r \to \infty$ is constructed. This basis $\mathfrak{h}^{\infty,\pm}$ captures the asymptotic behavior of any solution to ODE (3.5) as $r \to \infty$. In particular, as $r \to \infty$, $\mathfrak{h}^{\infty,+}$ grows exponentially and $\mathfrak{h}^{\infty,-}$ decays exponentially. It will be shown that after the addition of the pure gauge solution h_{pg} defined in equations (3.64) and (3.65), $h + h_{\text{pg}}$ is a mode solution in harmonic/transverse-traceless gauge (1.10) to the linearized Einstein vacuum equation, which is a linear combination of solutions that grow or decay exponentially as $r \to \infty$. The admissible boundary condition will be that the solution should decay exponentially, from which it will follows that $\mathfrak{h} = a\mathfrak{h}^{\infty,-}$.

One should note that the functions $P_k(r)$ and $Q_k(r) - \frac{\mu^2 r^2}{(r-2M)^2}$ admit convergent series expansions in a neighborhood of $r = \infty$,

$$P_k(r) = \sum_{n=0}^{\infty}\frac{p_n}{r^n}, \qquad Q_k(r) = \sum_{n=0}^{\infty}\frac{q_n}{r^n}, \tag{3.97}$$

with $p_0 = 0$, $p_1 = -4$, $q_0 = -(k^2 + \mu^2)$, and $q_1 = -2M(k^2 + 2\mu^2)$. Therefore, $r = \infty$ is an irregular singular point of ODE (3.5) according to the discussion of Appendix B. Eqs. (B18) and (B19) from Appendix B give

$$\lambda_\pm = \pm\sqrt{\mu^2 + k^2}, \quad \mu_\pm = 2 \pm \frac{M(2\mu^2 + k^2)}{\sqrt{\mu^2 + k^2}}. \tag{3.98}$$

From Theorem B.3, there exists a unique basis for solutions $\mathfrak{h}^{\infty,\pm}(r)$ to ODE (3.5) satisfying

$$\mathfrak{h}^{\infty,\pm} = e^{\pm\sqrt{\mu^2+k^2}r} r^{2\pm\frac{M(2\mu^2+k^2)}{\sqrt{\mu^2+k^2}}} + \mathcal{O}\left(e^{\pm\sqrt{\mu^2+k^2}r} r^{1\pm\frac{M(2\mu^2+k^2)}{\sqrt{\mu^2+k^2}}}\right). \tag{3.99}$$

Therefore, a general solution will be of the form

$$\mathfrak{h} = c_1\mathfrak{h}^{\infty,+} + c_2\mathfrak{h}^{\infty,-}, \tag{3.100}$$

with $c_1, c_2 \in \mathbb{R}$.

Proposition 3.7. Let $\mathfrak{h}$ be a solution to ODE (3.5). Let h be the mode solution to the linearized vacuum Einstein equation (2.4) in spherical gauge (3.2) associated with the solution $\mathfrak{h}$, and let h_{pg} be the pure gauge solution defined by Eqs. (3.64) and (3.65) such that $h + h_{\text{pg}}$ satisfies the harmonic/transverse-traceless gauge (1.10) conditions. Then, the solution $h + h_{\text{pg}}$ to ODE (3.5) decays exponentially towards spacelike infinity i_A^0 if $c_1 = 0$, where c_1 is defined by Eq. (3.100).

Proof. Defining $H_z^{\infty,\pm}(r) := \mathfrak{h}^{\infty,\pm}(r)$ and using Eqs. (3.66) and (3.67), one can construct the corresponding basis for solutions as $\tilde{H}_{tt}$, $\tilde{H}_{tr}$, $\tilde{H}_{rr}$, and $\tilde{H}_{\theta\theta}$ from Proposition 3.2. Note that Eqs. (3.66) and (3.67) define the components of the mode solution $h + h_{\text{pg}}$ to the linearized vacuum Einstein equation (2.4), which satisfies harmonic/transverse-traceless gauge (1.10). Asymptotically, $\tilde{H}_{tt}$, $\tilde{H}_{tr}$, $\tilde{H}_{rr}$, and $\tilde{H}_{\theta\theta}$ have the following behavior:

$$H_{tt}^{\infty,\pm} = e^{\pm\sqrt{\mu^2+k^2}r} r^{-1\pm\frac{M(2\mu^2+k^2)}{\sqrt{\mu^2+k^2}}} + \mathcal{O}\left(e^{\pm\sqrt{\mu^2+k^2}r} r^{-2\pm\frac{M(2\mu^2+k^2)}{\sqrt{\mu^2+k^2}}}\right), \tag{3.101}$$

$$H_{tr}^{\infty,\pm} = e^{\pm\sqrt{\mu^2+k^2}r} r^{-1\pm\frac{M(2\mu^2+k^2)}{\sqrt{\mu^2+k^2}}} + \mathcal{O}\left(e^{\pm\sqrt{\mu^2+k^2}r} r^{-2\pm\frac{M(2\mu^2+k^2)}{\sqrt{\mu^2+k^2}}}\right), \tag{3.102}$$

$$H_{rr}^{\infty,\pm} = e^{\pm\sqrt{\mu^2+k^2}r} r^{-1\pm\frac{M(2\mu^2+k^2)}{\sqrt{\mu^2+k^2}}} + \mathcal{O}\left(e^{\pm\sqrt{\mu^2+k^2}r} r^{-2\pm\frac{M(2\mu^2+k^2)}{\sqrt{\mu^2+k^2}}}\right), \tag{3.103}$$

$$H_{\theta\theta}^{\infty,\pm} = e^{\pm\sqrt{\mu^2+k^2}r} r^{1\pm\frac{M(2\mu^2+k^2)}{\sqrt{\mu^2+k^2}}} + \mathcal{O}\left(e^{\pm\sqrt{\mu^2+k^2}r} r^{\pm\frac{M(2\mu^2+k^2)}{\sqrt{\mu^2+k^2}}}\right). \tag{3.104}$$

It is clear from these expressions that if $c_1 = 0$, then the mode solution $h + h_{\text{pg}}$ decays exponentially as $r \to \infty$.

□

E. Reduction of the proof of theorem 1.1

This section summarizes Propositions 3.2–3.5 and 3.7 to give a full description of the permissible asymptotic behavior of a mode solution h in spherical gauge (3.2), which is not pure gauge. This provides a reduction of Theorem 1.1 to proving that there exists a solution $\mathfrak{h}$ to ODE (3.5), which has $\mu > 0$, $k \neq 0$, and obeys the admissible boundary conditions: $k_2 = 0$ and $c_1 = 0$.

Proposition 3.8. Let $\mu > 0$ and $k \in \mathbb{R}$ with $k \neq 0$. Let $\mathfrak{h}^{2M,\pm}$ be the basis for the space of solutions to ODE (3.5) as defined in Eqs. (3.41) and (3.42) and $\mathfrak{h}^{\infty,\pm}$ be the basis for the space of solutions to ODE (3.5) as defined in Eq. (3.99). In particular, to any solution $\mathfrak{h}$ of ODE (3.5), one can ascribe four numbers $k_1, k_2, c_1, c_2 \in \mathbb{R}$ defined by

$$\mathfrak{h}(r) = k_1\mathfrak{h}^{2M,+}(r) + k_2\mathfrak{h}^{2M,-}(r), \tag{3.105}$$

$$\mathfrak{h}(r) = c_1\mathfrak{h}^{\infty,+}(r) + c_2\mathfrak{h}^{\infty,-}(r). \tag{3.106}$$

Let h be the mode solution in spherical gauge (3.2) to the linearized vacuum Einstein Eq. (2.4) on the exterior $\mathcal{E}_A$ of Schwarzschild black string $\mathrm{Sch}_4 \times \mathbb{R}$ *associated with $\mathfrak{h}$ via Proposition 3.2. Let h_{pg} be the pure gauge solution as defined in Eqs. (3.64) and (3.65). Then, $h + h_{\text{pg}}$ decays exponentially towards spacelike infinity i_A^0 and is smooth at the future event horizon $\mathcal{H}_A^+$* if $k_2 = 0$ *and* $c_1 = 0$. *Moreover, $h + h_{\text{pg}}$ satisfies the harmonic/transverse-traceless gauge conditions (1.10) and cannot be a pure gauge solution.*

Under the additional assumption that $kR \in \mathbb{Z}$, the mode solution h defined above can be interpreted as a mode solution to the linearized vacuum Einstein equation (2.4) on the exterior $\mathcal{E}_A$ of the Schwarzschild black string $\mathrm{Sch}_4 \times \mathbb{S}^1_R$. Hence, if $kR \in \mathbb{Z}$, the above statement applies to the exterior $\mathcal{E}_A$ of $\mathrm{Sch}_4 \times \mathbb{S}^1_R$.

Section IV (see Proposition 4.1) will prove the existence of a solution $\mathfrak{h}$ to ODE (3.5) satisfying the properties of Proposition 3.8. In particular, for all $|k| \in [\frac{3}{20M}, \frac{8}{20M}]$, a solution $\mathfrak{h}$ to ODE (3.5) with $\mu > \frac{1}{40\sqrt{10}M} > 0$, $k_2 = 0$, and $c_1 = 0$ is constructed. If $R > 4M$, then there exists an integer $n \in [\frac{3R}{20M}, \frac{8R}{20M}]$. Hence, one can choose k such that the constructed $\mathfrak{h}$ gives rise to a mode solution on $\mathrm{Sch}_4 \times \mathbb{S}^1_R$. Moreover, on $\mathrm{Sch}_4 \times \mathbb{S}^1_R$, h will manifestly have finite energy in the sense that $\|h|_\Sigma\|_{H^1}$ and $\|\partial_{t_*} h|_\Sigma\|_{L^2}$ are finite. (Note that on $\mathrm{Sch}_4 \times \mathbb{R}$, h will not have finite energy due to the periodic behavior in z on $\mathbb{R}$.) Thus, Theorem 1.1 follows from Propositions 3.8 and 4.1.

IV. THE VARIATIONAL ARGUMENT

By Proposition 3.8, the Proof of Theorem 1.1 has now been reduced to exhibiting a solution $\mathfrak{h}$ to (3.5) with $\mu > 0$, $k \neq 0$, $k_2 = 0$, and $c_1 = 0$. This section establishes the required proposition thus completing the proof.

Proposition 4.1. For all $|k| \in [\frac{3}{20M}, \frac{8}{20M}]$, there exists a $C^\infty((2M,\infty))$ solution $\mathfrak{h}$ to ODE (3.5) with $\mu > 0$, and in the language of Proposition 3.8, $k_2 = 0$ and $c_1 = 0$.

In order to exhibit such a solution $\mathfrak{h}$ to ODE (3.5), it is convenient to rescale the solution and change coordinates in ODE (3.5) so as to recast as a Schrödinger equation for a function u. This transformation is given in Sec. IV A. In Sec. IV B, an energy functional is assigned to the resulting Schrödinger operator. With the use of a test function (constructed in Sec. IV C), a direct variational argument can be run to establish that for $|k| \in [\frac{3}{20M}, \frac{8}{20M}]$, there exists a weak solution $u \in H^1(\mathbb{R})$ with $\|u\|_{H^1(\mathbb{R})} = 1$ such that $\mu > 0$. The Proof of Proposition 4.1 concludes by showing that the solution u is indeed smooth for $r \in (2M,\infty)$ and satisfies the conditions of Proposition 3.8, i.e., $k_2 = 0$ and $c_1 = 0$.

A. Schrödinger reformulation

To reduce the number of parameters in ODE (3.5), one can eliminate the mass parameter with $x := \frac{r}{2M}$, $\hat{\mu} := 2M\mu$, and $\hat{k} := 2Mk$ to find

$$\frac{d^2\mathfrak{h}}{dx^2}(x) + p_{\hat{k}}(x)\frac{d\mathfrak{h}}{dx} + \left(q_{\hat{k}}(x) - \frac{\hat{\mu}^2 x^2}{(x-1)^2}\right)\mathfrak{h}(x) = 0, \tag{4.1}$$

with

$$p_{\hat{k}}(x) = \frac{1}{x-1} - \frac{5}{x} + \frac{6}{x(\hat{k}^2x^3+1)}, \tag{4.2}$$

$$q_{\hat{k}}(x) = \frac{3}{x^2(x-1)} - \frac{\hat{k}^2 x}{x-1} - \frac{3}{x^2(x-1)(1+\hat{k}^2x^3)}. \tag{4.3}$$

Following Proposition C.1 from Appendix C, one can now transform Eq. (4.1) into the regularized Schrödinger form by introducing a weight function $\mathfrak{h}(x) = w(x)\tilde{\mathfrak{h}}(x)$ and changing coordinates to $x_* = \frac{r_*}{2M} = x + \log|x-1|$. This will produce a Schrödinger operator with a potential, which decays to zero at the future event horizon and tends to the constant $\hat{k}^2$ at spatial infinity. From Proposition C.1, the weight function must satisfy the ODE,

$$\frac{dw}{dx} + \frac{(1-2k^2x^3)}{x(1+k^2x^3)}w = 0. \tag{4.4}$$

The desired solution for the weight function is

$$w(x) = \frac{(1+\hat{k}^2x^3)}{x}. \tag{4.5}$$

ODE (4.1) becomes

$$-\frac{d^2\tilde{\mathfrak{h}}}{dx_*^2}(x_*) + V(x_*)\tilde{\mathfrak{h}}(x_*) = -\hat{\mu}^2\tilde{\mathfrak{h}}(x_*), \tag{4.6}$$

where $V : \mathbb{R} \to \mathbb{R}$ can be found from Eq. (C10) to be

$$V(x_*) = \hat{k}^2\frac{(x-1)}{x} + \frac{(6x-11)(x-1)}{x^4} + \frac{18(x-1)^2}{x^4(1+\hat{k}^2x^3)^2} - \frac{6(4x-5)(x-1)}{x^4(1+\hat{k}^2x^3)}, \quad x \in (1,\infty), \tag{4.7}$$

where x is understood as an implicit function of x_*.

As a trivial consequence of Proposition 3.8 in Sec. III D on asymptotics of the solution to ODE (3.5), one has the following proposition for the asymptotics of the Schrödinger equation (4.6).

Proposition 4.2. Assume $\hat{\mu} > 0$. To any solution $\tilde{\mathfrak{h}}$ to the Schrödinger equation (4.6), one can ascribe four numbers $\tilde{k}_1, \tilde{k}_2, \tilde{c}_1, \tilde{c}_2 \in \mathbb{R}$ defined by

$$\tilde{\mathfrak{h}}(x_*) = \tilde{k}_1 \tilde{\mathfrak{h}}^{2M,+}(x_*) + \tilde{k}_2 \tilde{\mathfrak{h}}^{2M,-}(x_*) \quad \text{as} \quad x_* \to -\infty, \tag{4.8}$$

$$\tilde{\mathfrak{h}}(x_*) = \tilde{c}_1 \tilde{\mathfrak{h}}^{\infty,+}(x_*) + \tilde{c}_2 \tilde{\mathfrak{h}}^{\infty,-}(x_*) \quad \text{as} \quad x_* \to \infty, \tag{4.9}$$

with

$$\tilde{\mathfrak{h}}^{2M,\pm} := \frac{\mathfrak{h}^{2M,\pm}}{w}, \tag{4.10}$$

$$\tilde{\mathfrak{h}}^{\infty,\pm} := \frac{\mathfrak{h}^{\infty,\pm}}{w}. \tag{4.11}$$

The conditions that $\tilde{c}_1 = 0$ and $\tilde{k}_2 = 0$ are equivalent to, in the language of Proposition 3.8, $c_1 = 0$ and $k_2 = 0$.

Remark. In the case $4M\mu$ is not a positive integer or $4M\mu$ is a positive integer and $C_N = 0$, the leading order terms of these basis elements are

$$\tilde{\mathfrak{h}}^{2M,\pm} = (x-1)^{\pm\hat{\mu}} \left(\frac{1}{1+\hat{k}^2} + \mathcal{O}(x-1) \right), \tag{4.12}$$

$$\tilde{\mathfrak{h}}^{\infty,\pm} = e^{\pm\sqrt{\hat{\mu}^2+\hat{k}^2}x} x^{\pm\frac{(2\hat{\mu}^2+\hat{k}^2)}{2\sqrt{\hat{\mu}^2+\hat{k}^2}}} \left(\frac{1}{\hat{k}^2} + \mathcal{O}\left(\frac{1}{x}\right) \right). \tag{4.13}$$

B. Direct variational argument

This section establishes a variational argument which, will be used to infer the existence of a negative eigenvalue to the Schrödinger operator in Eq. (4.6).

Proposition 4.3. Let $W : \mathbb{R} \to \mathbb{R}$ and define

$$E_0 := \inf_{\substack{v \in H^1(\mathbb{R}) \\ \|v\|_{L^2(\mathbb{R})}=1}} \left\{ E(v) := \langle \nabla v, \nabla v \rangle_{L^2(\mathbb{R})} + \langle Wv, v \rangle_{L^2(\mathbb{R})} \right\}. \tag{4.14}$$

Suppose that $W = p + q$ with $q \in C^0(\mathbb{R})$ such that

$$\lim_{|x|\to\infty} q(x) = 0 \tag{4.15}$$

and $p(x) \in L^\infty(\mathbb{R})$ positive. If $E_0 < 0$, then there exists $u \in H^1(\mathbb{R})$ such that $\|u\|_{L^2(\mathbb{R})} = 1$ and $E(u) = E_0$.

Proof. By the definition of the infimum, there exists a minimizing sequence $(u_m)_m \subset H^1(\mathbb{R})$ and $\|u_m\|_{L^2} = 1$ such that

$$\lim_{n\to\infty} E(u_n) = E_0. \tag{4.16}$$

Now, u_n are bounded in $H^1(\mathbb{R})$ by the following argument. There exists an $M \in \mathbb{N}$ such that for all $m \geq M$,

$$E(u_m) \leq E_0 + 1. \tag{4.17}$$

Hence, for $m \geq M$,

$$\langle \nabla u_m, \nabla u_m \rangle_{L^2(\mathbb{R})} \leq E_0 + 1 + \sup_{x\in\mathbb{R}} |p(x)| + \sup_{x\in\mathbb{R}} |q(x)|. \tag{4.18}$$

Hence, $\|u_m\|_{H^1(\mathbb{R})}$ is controlled. Now, using Theorem D.1 from Appendix D, there exists a subsequence $(u_{m_n})_n$ such that $u_{m_n} \rightharpoonup u$ in $H^1(\mathbb{R})$.

Consider

$$E(u_m) = \int_{\mathbb{R}} |\nabla u_m|^2 + p(x)|u_m|^2 + q(x)|u_m|^2 dx. \tag{4.19}$$

Since the Dirichlet energy is lower semicontinuous, only the latter two terms under the integral (4.19) need to be examined more carefully. The middle term in integral (4.19) is simply a weighted L^2 integral, so lower semicontinuity is established via

$$\|u_n - u\|_{L_p^2}^2 = \langle u_n - u, u_n - u\rangle_{L_p^2} = \|u_n\|_{L_q^2}^2 - 2\langle u_n, u\rangle_{L_p^2} + \|u\|_{L_p^2}^2. \tag{4.20}$$

Hence,

$$\|u\|_{L_p^2}^2 \le \|u_n\|_{L_p^2}^2 - 2\langle u, u_n - u\rangle_{L_p^2}. \tag{4.21}$$

Hence, by weak convergence,

$$\|u\|_{L_p^2}^2 \le \liminf_{n\to\infty} \|u_n\|_{L_p^2}^2. \tag{4.22}$$

Proposition D.2 from Appendix D establishes that the multiplication operator $M_q : u \to qu$ is compact from $H^1(\mathbb{R})$ to $L^2(\mathbb{R})$. Hence, by the characterization of compactness through weak convergence (Theorem D.1 from Appendix D), $qu_m \to qu$ in $L^2(\mathbb{R})$. Therefore,

$$\langle qu, u\rangle_{L^2} = \lim_{m\to\infty} \langle qu_m, u_m\rangle_{L^2} = \liminf_{m\to\infty} \langle qu_m, u_m\rangle_{L^2}. \tag{4.23}$$

Hence, the last term under integral (4.19) is also lower semicontinuous. Therefore,

$$E(u) \le \liminf_{n\to\infty} E(u_n) = E_0. \tag{4.24}$$

Since the infimum is negative, the minimizer is non-trivial. One needs to show that there is no loss of mass, i.e., $\|u\|_{L^2} = 1$. Note $\|u\|_{L^2} \le \liminf_{n\to\infty} \|u_n\|_{L^2} = 1$. Hence, suppose $\|u\|_{L^2} < 1$ and define $\tilde{u} = \frac{u}{\|u\|_{L^2}}$ so $\|\tilde{u}\|_{L^2} = 1$, then

$$E(\tilde{u}) = \frac{E_0}{\|u\|_{L^2(\mathbb{R})}^2} \le E_0 \tag{4.25}$$

since $\|u\|_{L^2} \le 1$. Hence, one would obtain a contradiction to the infimum if $\|u\|_{L^2} < 1$.

□

Corollary 4.4. Let $W = V$ with V as defined in Eq. (4.7), then

$$E(v) := \langle \nabla v, \nabla v\rangle_{L^2(\mathbb{R})} + \langle Vv, v\rangle_{L^2(\mathbb{R})} \qquad E_0 := \inf_{\substack{v\in H^1(\mathbb{R})\\ \|v\|_{L^2(\mathbb{R})}=1}} E(v) \tag{4.26}$$

satisfies the assumptions of Proposition 4.6.

Proof. The function $V : \mathbb{R} \to \mathbb{R}$ can be written as $V = p + q$ with p and q as follows. Define

$$p(x_*) := \hat{k}^2 \frac{x-1}{x}, \tag{4.27}$$

$$q(x_*) := \frac{(6x-11)(x-1)}{x^4} + \frac{18(x-1)^2}{x^4(1+\hat{k}^2x^3)^2} - \frac{6(4x-5)(x-1)}{x^4(1+\hat{k}^2x^3)}, \tag{4.28}$$

where in these expressions x considered as an implicit function of x_*. Since $x \in (1,\infty)$, it follows that $p(x_*) > 0$ for all $x_* \in \mathbb{R}$. Moreover,

$$\sup_{x_*\in\mathbb{R}} |p(x_*)| = 1. \tag{4.29}$$

Therefore, $p \in L^\infty(\mathbb{R})$. Note that the function q satisfies $\lim_{|x_*|\to\infty} q(x_*) = 0$. Hence, the assumptions of Proposition 4.3 hold.

□

C. The test function and existence of a minimizer

ODE (4.6) is now in a form where a direct variational argument can be used to prove that there exists an eigenfunction of the Schrödinger operator associated with the left-hand side of ODE (4.6) with a negative eigenvalue, i.e., $-\hat{\mu}^2 < 0$. The following proposition constructs a suitable test function such that it is in the correct function space, $H^1(\mathbb{R})$, and, for all $|\hat{k}| \in [\frac{3}{10}, \frac{8}{10}]$, implies that the infimum of the energy functional in Eq. (4.26) is negative. (As will be apparent, the negativity is inferred via complicated but purely algebraic calculations.)

Proposition 4.5. Define $u_T(x_*) := x(1 + |\hat{k}|^2 x^3)(x-1)^{\frac{1}{n}} e^{-4|\hat{k}|(x-1)}$, *where x is an implicit function of x_*, n is a finite non-zero natural number, $\hat{k} \in \mathbb{R}\backslash\{0\}$ and define E and E_0 as in Eq. (4.26) of Corollary 4.4. Then, $u_T \in H^1(\mathbb{R})$ and for $n = 100$ and $|\hat{k}| \in [\frac{3}{10}, \frac{8}{10}]$, $E_0 \le \frac{E(u_T)}{\|u_T\|^2_{L^2(\mathbb{R})}} < -\frac{1}{4000}$.*

Proof. Let $k \in \mathbb{N} \cup \{0\}$ and define the following functions:

$$f_j(x) := x^{j-1}(x-1)^{\frac{2}{n}-1} e^{-8|\hat{k}|(x-1)}. \tag{4.30}$$

The $H^1(\mathbb{R})$-norm of u_T can be expressed as

$$\|u_T\|^2_{H^1(\mathbb{R})} = \int_1^\infty \left| \frac{x-1}{x} \frac{du_T}{dx} \right|^2 \frac{x}{x-1} dx + \int_1^\infty |u_T|^2 \frac{x}{x-1} dx, \tag{4.31}$$

where on the right-hand side the change of variables from $x_* \in \mathbb{R}$ to $x \in (1, \infty)$ has been made. To calculate $\|u_T\|_{L^2(\mathbb{R})}$, it is useful to write it as a linear combination of the functions f_k in Eq. (4.30). Explicitly, one can show that

$$|u_T|^2 \frac{x}{x-1} = f_4(x) + 2|\hat{k}|^2 f_7(x) + |\hat{k}|^4 f_{10}(x). \tag{4.32}$$

Similarly, one can show that

$$\left| \frac{x-1}{x} \frac{du_T}{dx} \right|^2 \frac{x}{x-1} = \sum_{j=1}^{11} c_j f_{j-1}(x), \tag{4.33}$$

with

$$\begin{aligned}
&c_1 = 1, \qquad c_2 = -2 - \frac{2}{n} - 8|\hat{k}|, \qquad c_3 = 1 + \frac{1}{n^2} + \frac{2}{n} + 16|\hat{k}| + \frac{8|\hat{k}|}{n} + 16|\hat{k}|^2,\\
&c_4 = -8|\hat{k}| - \frac{8|\hat{k}|}{n} - 24|\hat{k}|^2, \qquad c_5 = -\frac{10|\hat{k}|^2}{n} - 40|\hat{k}|^3,\\
&c_6 = 8|\hat{k}|^2 + \frac{2|\hat{k}|^2}{n^2} + \frac{10|\hat{k}|^2}{n} + 80|\hat{k}|^3 + \frac{16|\hat{k}|^3}{n} + 32|\hat{k}|^4, \qquad c_7 = -40|\hat{k}|^3 - \frac{16|\hat{k}|^3}{n} - 48|\hat{k}|^4,\\
&c_8 = -\frac{8|\hat{k}|^4}{n} - 32|\hat{k}|^5, \qquad c_9 = 16|\hat{k}|^4 + \frac{|\hat{k}|^4}{n^2} + \frac{8|\hat{k}|^4}{n} + 64|\hat{k}|^5 + \frac{8|\hat{k}|^5}{n} + 16|\hat{k}|^6,\\
&c_{10} = -32|\hat{k}|^5 - \frac{8|\hat{k}|^5}{n} - 32|\hat{k}|^6, \qquad c_{11} = 16|\hat{k}|^6.
\end{aligned} \tag{4.34}$$

One can express $E(u_T)$ with the change of variables from x_* to x as

$$E(u_T) = \int_1^\infty \left(\left| \frac{x-1}{x} \frac{du_T}{dx} \right|^2 + V(u_T)^2 \right) \frac{x}{x-1} dx. \tag{4.35}$$

The integrand can be written as

$$\left(\left| \frac{x-1}{x} \frac{du_T}{dx} \right|^2 + V(u_T)^2 \right) \frac{x}{x-1} = \sum_{j=1}^{11} a_j f_{j-1}(x), \tag{4.36}$$

with

$$a_1 = 0, \quad a_2 = -\frac{2+n+8n|\hat{k}|}{n}, \quad a_3 = 1 + \frac{1}{n^2} + 16|\hat{k}| + 16|\hat{k}|^2 + \frac{2+8|\hat{k}|}{n},$$
$$a_4 = -|\hat{k}|\left(\frac{8(1+n)}{n} + 33|\hat{k}|\right), \quad a_5 = \frac{(21n-10)|\hat{k}|^2}{n} - 40|\hat{k}|^3,$$
$$a_6 = |\hat{k}|^2\left(\frac{2(1+5n-2n^2)}{n^2} + \frac{16(1+5n)|\hat{k}|}{n} + 32|\hat{k}|^2\right), \quad a_7 = -|\hat{k}|^3\left(\frac{8(2+5n)}{n} + 39|\hat{k}|\right), \tag{4.37}$$
$$a_8 = -|\hat{k}|^4\left(\frac{(8+15n)}{n} + 32|\hat{k}|\right), \quad a_9 = |\hat{k}|^4\left(\frac{1+8n+22n^2}{n^2} + \frac{8(1+8n)}{n} + 16|\hat{k}|^2\right),$$
$$a_{10} = -|\hat{k}|^5\left(\frac{8(1+4n)}{n} + 33|\hat{k}|\right), \quad a_{11} = 17|\hat{k}|^6.$$

Therefore, if one can compute the integrals

$$I_j := \int_1^\infty f_j(x)dx \tag{4.38}$$

for $k = 0, \ldots, 10$, then one can compute $\|u_T\|_{L^2(\mathbb{R})}$, $\|\frac{du_T}{dx_*}\|_{L^2(\mathbb{R})}$, and $E(u_T)$.

Defining a change variables in the integrals (4.38) by $t = x - 1$, integrals (4.38) become

$$I_j = \int_0^\infty (t+1)^{j-1} t^{\frac{2}{n}-1} e^{-8|\hat{k}|t}. \tag{4.39}$$

Note that the confluent hypergeometric function of the second kind $U(a, b; z)$ can be defined as

$$U(a,b;z) := \frac{1}{\Gamma(a)} \int_0^\infty (t+1)^{b-a-1} t^{a-1} e^{-zt} \tag{4.40}$$

for $a, b, z \in \mathbb{C}$ with $\mathrm{Re}(a) > 0$ and $\mathrm{Re}(z) > 0$, where $\Gamma(a)$ is the Euler Gamma function, which can be defined through the integral

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt \tag{4.41}$$

for $a \in \mathbb{C}$ with $\mathrm{Re}(a) > 0$. For a reference, see chapter 9 of Ref. 27. Therefore, setting $a = \frac{2}{n}$, $b = k + \frac{2}{n}$, and $z = 8|\hat{k}|$ gives

$$I_j = \Gamma\left(\frac{2}{n}\right) U\left(\frac{2}{n}, j + \frac{2}{n}; 8|\hat{k}|\right). \tag{4.42}$$

The function $U(a, b; z)$ satisfies the following recurrence properties (see Chap. 9 of Ref. 27 and Chap. 16 of Ref. 28):

$$U(0,b;z) = 1, \tag{4.43}$$
$$U(a,b;z) - z^{1-b} U(1+a-b, 2-b; z) = 0, \tag{4.44}$$
$$U(a,b;z) - aU(a+1,b;z) - U(a,b-1;z) = 0, \tag{4.45}$$
$$(b-a-1)U(a,b-1;z) + (1-b-z)U(a,b;z) + zU(a,b+1;z) = 0. \tag{4.46}$$

Setting $a = \frac{2}{n}$, $b = 1 + \frac{2}{n}$, and $z = 8|\hat{k}|$ in Eq. (4.44) and using Eq. (4.43) allow one to calculate I_1. Setting $a = \frac{2}{n}$, $b = 2 + \frac{2}{n}$, and $z = 8|\hat{k}|$ in Eq. (4.45) and using I_1 and Eq. (4.43) allow one to calculate I_2. Setting $a = \frac{2}{n}$, $b = j + \frac{2}{n}$, and $z = 8|\hat{k}|$ in Eq. (4.46) and using $I_{j-1}, \ldots, I_1$ and Eq. (4.43) allow one to calculate I_j. Finally, one can show that $I_0 < \infty$ by the following argument. One can see from the definition of I_j in Eq. (4.39) that

$$I_0 = \int_1^\infty \frac{1}{x(x-1)} (x-1)^{\frac{2}{n}} e^{-8|\hat{k}|(x-1)} dx. \tag{4.47}$$

Now, since $e^{-8|\hat{k}|(x-1)} < 1$ on $x \in (1, \infty)$ and $\frac{(x-1)^{\frac{2}{n}-1}}{x} < \frac{1}{2}(x-1)$ for $n \geq 1$ on $x \in (2, \infty)$,

$$I_0 \leq \int_1^2 \frac{1}{x(x-1)} (x-1)^{\frac{2}{n}} + \frac{1}{2} \int_2^\infty (x-1) e^{-8|\hat{k}|(x-1)} < \infty. \tag{4.48}$$

Using the recurrence properties in Eqs. (4.43)–(4.46) and estimate (4.48) allows one to explicitly show that $\|u_T\|_{H^1(\mathbb{R})} < \infty$ for $n \geq 1$, $\hat{k} \in \mathbb{R}\backslash\{0\}$, i.e., $u_T \in H^1(\mathbb{R})$. Moreover, one can calculate $\frac{E(u_T)}{\|u_T\|_{L^2(\mathbb{R})}}$. Explicitly, $\frac{E(u_T)}{\|u_T\|_{L^2(\mathbb{R})}}$ is given by

$$\frac{E(u_T)}{\|u_T\|^2_{L^2(\mathbb{R})}} = \frac{|\hat{k}|^2 \sum_{i=1}^{9} p_i(n) |\hat{k}|^{i-1}}{\sum_{j=1}^{10} q_j(n) |\hat{k}|^{i-1}}, \tag{4.49}$$

with

$$\begin{aligned}
p_1(n) &:= 16 + 416n + 5576n^2 + 36176n^3 + 123809n^4 + 234794n^5 + 244459n^6 + 128034n^7 + 25560n^8,\\
p_2(n) &:= 32n(16 + 336n + 3296n^2 + 15572n^3 + 29107n^4 + 21238n^5 + 4361n^6 - 366n^7),\\
p_3(n) &:= 128n^2(56 + 924n + 6130n^2 + 20133n^3 + 11972n^4 - 3365n^5 - 466n^6),\\
p_4(n) &:= 1024n^3(56 + 700n + 2750n^2 + 6041n^3 - 1715n^4 - 18n^5),\\
p_5(n) &:= 2048n^4(140 + 1260n + 2225n^2 + 3443n^3 - 1758n^4),\\
p_6(n) &:= 32768n^5(28 + 168n + 43n^2 + 111n^3),\\
p_7(n) &:= 917504n^6(2 + 7n - 3n^2),\\
p_8(n) &:= 1048576n^7(2 + 3n),\\
p_9(n) &:= 1048576n^8,\\
q_1(n) &:= 116 + 288n + 2184n^2 + 9072n^3 + 22449n^4 + 33642n^5 + 29531n^6 + 13698n^7 + 2520n^8,\\
q_2(n) &:= 4n(144 + 2016n + 12104n^2 + 39120n^3 + 71801n^4 + 73494n^5 + 38171n^6 + 7590n^7),\\
q_3(n) &:= 128n^2(72 + 756n + 3534n^2 + 8535n^3 + 11180n^4 + 7137n^5 + 1642n^6),\\
q_4(n) &:= 1536n^3(56 + 420n + 1510n^2 + 2535n^3 + 2351n^4 + 706n^5),\\
q_5(n) &:= 2048n^4(252 + 1260n + 3485n^2 + 3495n^3 + 2554n^4),\\
q_6(n) &:= 8192n^5(252 + 756n + 1653n^2 + 669n^3 + 512n^4),\\
q_7(n) &:= 393216n^6(14 + 21n + 39n^2),\\
q_8(n) &:= 524288n^7(18 + 9n + 16n^2),\\
q_9(n) &:= 9437184n^8,\\
q_{10}(n) &:= 4194304n^9.
\end{aligned} \tag{4.50}$$

Taking $n = 100$, one can check, via Sturm's algorithm,[29] that the polynomial

$$\mathfrak{p}(n, |\hat{k}|) := \sum_{i=1}^{9} p_i(n) |\hat{k}|^{i-1} \tag{4.51}$$

has two distinct real roots in $|\hat{k}| \in (0, 1)$. Evaluating $\mathfrak{p}(100, |\hat{k}|)$ at $|\hat{k}| = 0$, $|\hat{k}| = \frac{3}{10}$, $|\hat{k}| = \frac{8}{10}$, and $|\hat{k}| = 1$ yields

$$\mathfrak{p}(100, 0) > 0, \quad \mathfrak{p}\left(100, \frac{3}{10}\right) < 0, \quad \mathfrak{p}\left(100, \frac{8}{10}\right) < 0 \quad \text{and} \quad \mathfrak{p}(100, 1) > 0. \tag{4.52}$$

Hence, $\frac{E(u_T)}{\|u_T\|^2_{L^2(\mathbb{R})}}$ must be negative for all $|\hat{k}| \in [\frac{3}{10}, \frac{8}{10}]$. Taking the derivative of $\frac{E(u_T)}{\|u_T\|^2_{L^2(\mathbb{R})}}$ with respect to $|\hat{k}|$ yields another rational function of $|\hat{k}|$ with the positive denominator. Evaluating at the end points of the interval $|\hat{k}| \in [\frac{3}{10}, \frac{8}{10}]$ yields $\frac{d}{d|\hat{k}|}\left(\frac{E(u_T)}{\|u_T\|^2_{L^2(\mathbb{R})}}\right)|_{n=100} < 0$ at $|\hat{k}| = \frac{3}{10}$ and $\frac{d}{d|\hat{k}|}\left(\frac{E(u_T)}{\|u_T\|^2_{L^2(\mathbb{R})}}\right)|_{n=100} > 0$ at $|\hat{k}| = \frac{8}{10}$. Using Sturm's algorithm once again, one can check that the numerator of $\frac{d}{d|\hat{k}|}\left(\frac{E(u_T)}{\|u_T\|^2_{L^2(\mathbb{R})}}\right)$ has one distinct root in $|\hat{k}| \in (\frac{3}{10}, \frac{8}{10})$. Hence, $\frac{E(u_T)}{\|u_T\|^2_{L^2(\mathbb{R})}}$ with $|\hat{k}| \in [\frac{3}{10}, \frac{8}{10}]$ attains its maximum in at one of the end points. Further evaluating $\frac{E(u_T)}{\|u_T\|^2_{L^2(\mathbb{R})}}|_{n=100}$ at the end points of the interval $|\hat{k}| \in [\frac{3}{10}, \frac{8}{10}]$, one finds

$$\frac{E(u_T)}{\|u_T\|^2_{L^2(\mathbb{R})}}|_{n=100} < -\frac{1}{4000} \tag{4.53}$$

for all $|\hat{k}| \in [\frac{3}{10}, \frac{8}{10}]$. Hence, $E_0 \le \frac{E(u_T)}{\|u_T\|^2_{L^2(\mathbb{R})}}|_{n=100} < -\frac{1}{4000} < 0$ for all $|\hat{k}| \in [\frac{3}{10}, \frac{8}{10}]$.

□

D. Proof of proposition 4.1

To prove Proposition 4.1, one can clearly reformulate as follows:

Proposition 4.6. For all $|\hat{k}| \in [\frac{3}{10}, \frac{8}{10}]$, there exists a $C^\infty(\mathbb{R})$ solution $\tilde{\mathfrak{h}}$ to the Schrödinger equation (4.6) with $\hat{\mu} > \frac{1}{20\sqrt{10}} > 0$, and in the language of Proposition 4.2, $\tilde{k}_2 = 0$ and $\tilde{c}_1 = 0$.

Proof. By Proposition 4.3, Corollary 4.4, and Proposition 4.5, for all $k \in [\frac{3}{10}, \frac{8}{10}]$, there exists a minimizer $u \in H^1(\mathbb{R})$ with $\|u\|_{L^2(\mathbb{R})} = 1$ such that

$$E(u) = E_0 := \inf\{\langle \nabla v, \nabla v\rangle_{L^2(\mathbb{R})} + \langle Vv, v\rangle_{L^2(\mathbb{R})} : v \in H^1(\mathbb{R}), \|v\|_{L^2(\mathbb{R})} = 1\}, \tag{4.54}$$

with V as defined in Eq. (4.7). Moreover, by Proposition 4.5, $E_0 < -\frac{1}{4000} < 0$.

By standard Euler–Lagrange methods (see Theorem 3.21 and Example 3.22 in Ref. 30), u will weakly solve the ODE

$$-\frac{d^2u}{dx_*^2} + V(x_*)u = -\hat{\mu}^2 u, \tag{4.55}$$

with $-\hat{\mu}^2 = E_0$. From Proposition 4.5, $\hat{\mu}^2 = -E_0 > \frac{1}{4000}$. Hence, for all $|\hat{k}| \in [\frac{3}{10}, \frac{8}{10}]$, there exists a weak solution $u \in H^1(\mathbb{R})$ to the Schrödinger equation (4.6) with $\|u\|_{L^2(\mathbb{R})} = 1$ and $\hat{\mu} = \sqrt{-E_0} > \frac{1}{20\sqrt{10}}$.

From the regularity Theorem D.3, any $u \in H^1(\mathbb{R})$, which weakly solves the Schödinger equation (4.6), is, in fact, smooth. Therefore, for all $|\hat{k}| \in [\frac{3}{10}, \frac{8}{10}]$, there exists a solution $u \in C^\infty(\mathbb{R})$ to the Schrödinger equation (4.6) with $\hat{\mu} = \sqrt{-E_0} > \frac{1}{20\sqrt{10}}$.

To verify the boundary conditions of u, recall by Proposition 4.2, the solution u can be expressed, in the bases associated with $r = 2M$ and $r \to \infty$, as

$$u = \tilde{k}_1 \tilde{\mathfrak{h}}^{2M,+} + \tilde{k}_2 \tilde{\mathfrak{h}}^{2M,-}, \tag{4.56}$$

$$u = \tilde{c}_1 \tilde{\mathfrak{h}}^{\infty,+} + \tilde{c}_2 \tilde{\mathfrak{h}}^{\infty,-}, \tag{4.57}$$

with $\tilde{k}_1, \tilde{k}_2, \tilde{c}_1, \tilde{c}_2 \in \mathbb{R}$. Note that

$$\int_{-\infty}^{0} |\tilde{\mathfrak{h}}^{2M,-}|^2 dx_* = \int_1^{\frac{3}{2}} |\tilde{\mathfrak{h}}^{2M,-}|^2 \frac{x}{x-1} dx = \infty, \tag{4.58}$$

while

$$\int_{-\infty}^{0} |\tilde{\mathfrak{h}}^{2M,+}|^2 + |\Delta_{x_*}\mathfrak{h}^{2M,+}|^2 dx_* = \int_1^{\frac{3}{2}} \left(|\tilde{\mathfrak{h}}^{2M,+}|^2 + \left|\frac{x-1}{x}\Delta_x \mathfrak{h}^{2M,+}\right|^2\right) \frac{x}{x-1} dx < \infty. \tag{4.59}$$

Similarly, for $X_* > 0$ sufficiently large,

$$\int_{X_*}^{\infty} |\tilde{\mathfrak{h}}^{\infty,+}|^2 dx_* = \int_{x(X_*)}^{\infty} |\tilde{\mathfrak{h}}^{\infty,+}|^2 \frac{x}{x-1} dx = \infty, \tag{4.60}$$

while

$$\int_{X_*}^{\infty} |\tilde{\mathfrak{h}}^{\infty,-}|^2 + |\Delta_{x_*}\mathfrak{h}^{\infty,-}|^2 dx_* = \int_{x(X_*)}^{\infty} \left(|\tilde{\mathfrak{h}}^{\infty,-}|^2 + \left|\frac{x-1}{x}\Delta_x \mathfrak{h}^{\infty,-}\right|^2\right) \frac{x}{x-1} dx < \infty. \tag{4.61}$$

Therefore, since $u \in H^1(\mathbb{R})$, the solution u, in the language of Proposition 4.2, must have $\tilde{k}_2 = 0$ and $\tilde{c}_1 = 0$.

Therefore, taking $\tilde{\mathfrak{h}} = u$ and $|\hat{k}| \in [\frac{3}{10}, \frac{8}{10}]$ gives a $C^\infty(\mathbb{R})$ solution to the Schrödinger equation (4.6) with $\hat{\mu} > \frac{1}{20\sqrt{10}} > 0$, $\tilde{k}_2 = 0$, and $\tilde{c}_1 = 0$.

□

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APPENDIX A: CHRISTOFFEL AND RIEMANN TENSOR COMPONENTS FOR THE 5D SCHWARZSCHILD BLACK STRING

To compute $\Box_g h_{ab}$, one requires the Christoffel symbols and the Riemann tensor components; the non-zero Christoffel symboles are listed as follows:

$$\Gamma^t_{tr} = \frac{M}{r(r-2M)}, \tag{A1}$$

$$\Gamma^r_{tt} = \frac{M(r-2M)}{r^3}, \qquad \Gamma^r_{rr} = \frac{-M}{r(r-2M)}, \qquad \Gamma^r_{\theta\theta} = (2M-r), \qquad \Gamma^r_{\varphi\varphi} = (2M-r)\sin^2\theta, \tag{A2}$$

$$\Gamma^\theta_{r\theta} = \frac{1}{r} = \Gamma^\varphi_{r\varphi}, \qquad \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta, \qquad \Gamma^\varphi_{\theta\varphi} = \cot\theta. \tag{A3}$$

The others are obtained from symmetry of lower indices. Note, $R^z{}_{\mu\alpha\beta} = R^\mu{}_{z\alpha\beta} = R^\mu{}_{\alpha z\beta} = R^\mu{}_{\alpha\beta z} = 0$. Hence, the Riemann tensor components that are relevant are the ones with spacetime indices $\mu \in \{0,\ldots,3\}$, which are just the usual Schwarzschild Riemann tensor components; the non-zero ones are listed below for completeness,

$$R^t{}_{rtr} = \frac{2M}{r^2(r-2M)}, \qquad R^t{}_{\theta t\theta} = -\frac{M}{r}, \qquad R^t{}_{\varphi t\varphi} = -\frac{M\sin^2\theta}{r}, \tag{A4}$$

$$R^r{}_{trt} = -\frac{2M(r-2M)}{r^4}, \qquad R^r{}_{\theta r\theta} = -\frac{M}{r}, \qquad R^r{}_{\varphi r\varphi} = -\frac{M\sin^2\theta}{r}, \tag{A5}$$

$$R^\theta{}_{t\theta t} = \frac{M(r-2M)}{r^4}, \qquad R^\theta{}_{r\theta r} = -\frac{M}{r^2(r-2M)}, \qquad R^\theta{}_{\varphi\theta\varphi} = \frac{2M\sin^2\theta}{r}, \tag{A6}$$

$$R^\varphi{}_{t\varphi t} = \frac{M(r-2M)}{r^4}, \qquad R^\varphi{}_{r\varphi r} = -\frac{M}{r^2(r-2M)}, \qquad R^\varphi{}_{\theta\varphi\theta} = \frac{2M}{r}. \tag{A7}$$

Any others can be found from the $R^a{}_{b(cd)} = 0$ symmetry.

APPENDIX B: SINGULARITIES IN SECOND ORDER ODE

This section is heavily based on the book of Olver.[31] In particular, see Chap. 5 Secs. IV and V and Chap. 7 Sec. II.

Definition B.1 (Ordinary Point/Regular Singularity/Irregular Singularity). Let p and q be meromorphic functions on a subset of $\mathbb{C}$. Consider the linear second order ODE

$$\frac{d^2 f}{dz^2} + p(z)\frac{df}{dz} + q(z)f = 0. \tag{B1}$$

Then, $z_0 \in \mathbb{C}$ is an ordinary point of this differential equation if both $p(z)$ and $q(z)$ are analytic there. If z_0 is not an ordinary point and both

$$(z-z_0)p(z) \quad \text{and} \quad (z-z_0)^2 q(z) \tag{B2}$$

are analytic at z_0, then z_0 is a regular singularity; otherwise, z_0 is an irregular singularity.

Remark. The singular behavior of $z=\infty$ is determined by making the change of variables $\tilde{z} = \frac{1}{z}$ in ODE (B1). This case will be considered explicitly in Appendix B 2.

In the following, general results for ODE are presented.

1. Regular singularities

In this paper, solutions of a second order ODE in a neighborhood $|z - z_0| < r$ of a regular singular point are required. The classical method is to search for a convergent series solution in such a neighborhood.

Definition B.2 (Indicial Equation). Let p and q be meromorphic functions on a subset of $\mathbb{C}$. Consider the following second-order ODE with a regular singularity at $z_0 \in \mathbb{C}$,

$$\frac{d^2 f}{dz^2}(z) + p(z)\frac{df}{dz}(z) + q(z)f(z) = 0. \tag{B3}$$

Assume that there exist a convergent power series,

$$(z - z_0)p(z) = \sum_{j=0}^{\infty} p_j (z - z_0)^j, \qquad (z - z_0)^2 q(z) = \sum_{j=0}^{\infty} q_j (z - z_0)^j \qquad \forall |z - z_0| < r. \tag{B4}$$

The indicial equation is defined as

$$I(\alpha) := \alpha(\alpha - 1) + p_0\alpha + q_0 = 0. \tag{B5}$$

Remark. The indicial equation arises by considering the a solution of the form $f(z) = (z - z_0)^\alpha$ to the ODE

$$\frac{d^2 f}{dz^2}(z) + \frac{p_0}{z - z_0}\frac{df}{dz}(z) + \frac{q_0}{(z - z_0)^2}f(z) = 0. \tag{B6}$$

ODE (B6) is the leading order approximation of ODE (B3). The function $f(z) = (z - z_0)^\alpha$ solves ODE (B6) if the α satisfies the indicial equation.

The following two theorems deal with the asymptotic behavior of solutions in the neighborhood of a regular singularity.

Theorem B.1 (Frobenius). *Let p and q be meromorphic functions on a subset of $\mathbb{C}$. Consider the following second-order ODE with a regular singularity at $z_0 \in \mathbb{C}$:*

$$\frac{d^2 f}{dz^2}(z) + p(z)\frac{df}{dz}(z) + q(z)f(z) = 0, \tag{B7}$$

where

$$(z - z_0)p(z) = \sum_{j=0}^{\infty} p_j (z - z_0)^j, \qquad (z - z_0)^2 q(z) = \sum_{j=0}^{\infty} q_j (z - z_0)^j \tag{B8}$$

converge for all $|z - z_0| < r$, where $r > 0$. Let $\alpha_\pm$ be the two roots of the indicial equation. Suppose further that $\alpha_- \neq \alpha_+ + s$, where $s \in \mathbb{Z}$. Then, there exists a basis of solution to ODE (B7) of the form

$$f^+(z) = (z - z_0)^{\alpha_+}\sum_{j=0}^{\infty} a_j^+ (z - z_0)^j, \qquad f^-(z) = (z - z_0)^{\alpha_-}\sum_{j=0}^{\infty} a_j^- (z - z_0)^j, \tag{B9}$$

where these series converge for all z such that $|z - z_0| < r$. Moreover, a_j^+ and a_j^- can be calculated recursively by the formula

$$I(\alpha_\pm + j)a_j^\pm + (1 - \delta_{j,0})\sum_{s=0}^{j-1}\big((\alpha_\pm + s)p_{j-s} + q_{j-s}\big)a_s^\pm = 0. \tag{B10}$$

Remark. If the roots of the indicial equation do not differ by an integer, then Theorem B.1 gives a basis of solutions for the ODE in a neighborhood of the singular point. Equation (B10) determines the coefficients of the series expansion recursively from an arbitrarily assigned $a_0 \neq 0$, which can be taken to be 1. This process runs into difficulty if, and only if, the two roots differ by a positive integer. To see this, let α_+ be the root of the indicial equation with the largest real part, the other root is then $\alpha_+ - s$ for some $s \in \mathbb{Z}_+$. Then, since $I\big((\alpha_+ - s) + s\big) = 0$, one cannot determine a_s via Eq. (B10) for this power series. In this case, one solution can be found with the above method by taking the root of the indicial equation with the largest real part.

The following theorem investigates the case where the roots differ by an integer. Let α_+ be the root of the indicial equation with the largest real part, and the other root is then $\alpha_+ - s$ for some $s \in \mathbb{Z}_+ \cup \{0\}$.

Theorem B.2. *Consider ODE (B7) as in Theorem B.1 again satisfying (B8). Let α_+ and $\alpha_- = \alpha_+ - N$, with $N \in \mathbb{Z}_+ \cup \{0\}$, be roots of the indicial equation. Then, there exists a basis of solutions of the form*

$$f^+(z) = (z - z_0)^{\alpha_+} \sum_{j=0}^{\infty} a_j^+ (z - z_0)^j, \quad f^-(z) = (z - z_0)^{\gamma} \sum_{j=0}^{\infty} a_j^+ (z - z_0)^j + C_N f^+(z) \ln(z - z_0), \tag{B11}$$

with $\gamma = \alpha_+ + 1$ if $N = 0$ and $\gamma = \beta_-$ if $N \neq 0$, where these power series are convergent for all z such that $|z - z_0| < r$. Moreover, the coefficients a_j^+, a_j^-, and C_N can be calculated recursively.

2. Irregular singularities

This section summarizes the key result for constructing a basis of solutions to ODE (3.5) associated with $r \to \infty$. [The results presented can, in fact, be applied to any irregular singular point of an ODE (B1) since without loss of generality, the irregular singularity can be assumed to be at infinity after a change of coordinates.] The following definition makes precise the notion of an irregular singularity at infinity.

Definition B.3 (Irregular Singularity at Infinity). Let p and q be meromorphic functions on a subset of $\mathbb{C}$, which includes the set $\{z \in \mathbb{C} : |z| > a\}$. Consider the following second-order ODE

$$\frac{d^2 f}{dz^2} + p(z)\frac{df}{dz} + q(z) f = 0. \tag{B12}$$

Assume that for $|z| > a$, p and q may be expanded as convergent power series,

$$p(z) = \sum_{n=0}^{\infty} \frac{p_n}{z^n}, \qquad q(z) = \sum_{n=0}^{\infty} \frac{q_n}{z^n}. \tag{B13}$$

ODE (B12) has an irregular singular point at infinity if one of p_0, q_0, and q_1 does not vanish.

The main Theorem B.3 of this section can be motivated by the following discussion. Consider a formal power series

$$w = e^{\lambda z} z^{\mu} \sum_{n=0}^{\infty} \frac{a_n}{z^n}. \tag{B14}$$

Substituting the expansions into the ODE and equating coefficients yield

$$\lambda^2 + p_0 \lambda + q_0 = 0, \tag{B15}$$
$$(p_0 + 2\lambda)\mu = -(p_1 \lambda + q_1), \tag{B16}$$

and

$$(p_0 + 2\lambda) n a_n = (n - \mu)(n - 1 - \mu) a_{n-1} + \sum_{j=1}^{n} (\lambda p_{j+1} + q_{j+1} - (j - n - \mu) p_j) a_{n-j}. \tag{B17}$$

Now, Eq. (B15) has two roots,

$$\lambda_\pm = \frac{1}{2}\left(-p_0 \pm \sqrt{p_0^2 - 4g_0}\right). \tag{B18}$$

These give rise to

$$\mu_\pm = -\frac{p_1 \lambda_\pm + q_1}{p_0 + 2\lambda_\pm}. \tag{B19}$$

The two values of a_0, $a_0^\pm$ can be, without loss of generality, set to 1, and the higher order coefficients determined iteratively from Eq. (B17) unless one is in the exceptional case where $p_0^2 = 4g_0$ (for further information on this case, see Sec. I C of Chap. 7 in Ref. 31). The issue that arises is that in most cases, the formal series solution (B14) does not converge. However, the following theorem characterizes when (B14) provides an asymptotic expansion for the solution for sufficiently large $|z|$.

Theorem B.3. *Let $p(z)$ and $q(z)$ be meromorphic functions with convergent series expansions*

$$p(z) = \sum_{n=0}^{\infty} \frac{p_n}{z^n}, \qquad q(z) = \sum_{n=0}^{\infty} \frac{q_n}{z^n} \tag{B20}$$

for $|z| > a$ with $p_0^2 \neq 4q_0$. Then, the second order ODE

$$\frac{d^2 f}{dz^2} + p(z)\frac{df}{dz} + q(z) f = 0 \tag{B21}$$

has unique solutions $f^{\pm}(z)$ such that in the regions

$$\begin{cases} \{|z| > a\} \cap \{|\mathrm{Arg}((\lambda_- - \lambda_+)z)| \le \pi\} & (\text{for } f^+), \\ \{|z| > a\} \cap \{|\mathrm{Arg}((\lambda_+ - \lambda_-)z)| \le \pi\} & (\text{for } f^-) \end{cases} \tag{B22}$$

of the complex plane, $f^{\pm}$ is holomorphic, where $\lambda_{\pm}$ and $\mu_{\pm}$ are defined in Eqs. (B18) and (B19). Moreover, for all $N > 1$, $f^{\pm}(z)$ satisfies

$$f^{\pm}(z) = e^{\lambda_{\pm} z} z^{\mu_{\pm}} \left(\sum_{n=0}^{N-1} \frac{a_n^{\pm}}{z^n} + \mathcal{O}\left(\frac{1}{z^N}\right)\right) \tag{B23}$$

in the regions given in Eq. (B22).

APPENDIX C: TRANFORMATION TO SCHRÖDINGER FORM

Proposition C.1. Consider the second order homogeneous linear ODE

$$\frac{d^2 u}{dr^2} + p(r)\frac{du}{dr} + q(r)u = 0, \qquad p, q \in C^1(I), \qquad I \subset \mathbb{R}. \tag{C1}$$

Suppose that there exists a sufficiently regular coordinate transformation $s(r)$ and a function $w(r)$ such that

$$\frac{dw}{dr} + \frac{1}{2}\left(\frac{1}{\left(\frac{ds}{dr}\right)} \frac{d^2 s}{dr^2} + p \right) w = 0. \tag{C2}$$

Then, ODE (C1) can be reduced to the form

$$-\frac{d^2 z}{ds^2}(s) + V(s) z(s) = 0, \tag{C3}$$

with

$$V(s) = \frac{1}{2\left(\frac{ds}{dr}\right)^2}\left(\frac{dp}{dr} - \frac{3}{2\left(\frac{ds}{dr}\right)^2}\left(\frac{d^2 s}{dr^2}\right)^2 + \frac{1}{\left(\frac{ds}{dr}\right)}\frac{d^3 s}{dr^3} + \frac{p^2}{2} - 2g \right). \tag{C4}$$

Proof. The proof is a straight-forward calculation. Take $u(s) = w(s)z(s)$, then

$$\left(\frac{ds}{dr}\right)^2 w \frac{d^2 z}{ds^2} + \left(2\left(\frac{ds}{dr}\right)^2 \frac{dw}{ds} + w\frac{d^2 s}{dr^2} + pw\frac{ds}{dr}\right)\frac{dz}{ds} + \left(\left(\frac{ds}{dr}\right)^2 \frac{d^2 w}{ds^2} + \frac{dw}{ds}\frac{d^2 s}{dr^2} + p\frac{dw}{ds}\frac{ds}{dr} + qw \right) z = 0.$$

To reduce this to symmetric form, one can set

$$2\left(\frac{ds}{dr}\right)^2 \frac{dw}{ds} + w\frac{d^2 s}{dr^2} + pw\frac{ds}{dr} = 0, \tag{C5}$$

which is equivalent to $w(r)$ satisfying

$$\frac{dw}{dr} + \frac{1}{2}\left(\frac{1}{\left(\frac{ds}{dr}\right)} \frac{d^2 s}{dr^2} + p \right) w = 0. \tag{C6}$$

Hence,

$$\frac{d^2w}{dr^2} = -\frac{1}{2}\left(\frac{df}{dr} - \frac{1}{\left(\frac{ds}{dr}\right)^2}\left(\frac{d^2s}{dr^2}\right)^2 + \frac{1}{\left(\frac{ds}{dr}\right)}\frac{d^3s}{dr^3} - \frac{1}{2}\left(\frac{1}{\left(\frac{ds}{dr}\right)}\frac{d^2s}{dr^2} + p\right)^2\right)w. \tag{C7}$$

Note the last term in the ODE for z reduces to

$$\frac{d^2w}{dr^2} + p\frac{dw}{dr} + qw. \tag{C8}$$

Reducing this with the expressions for the derivatives of w gives the potential for $-\frac{d^2z}{ds^2} + V(s)z = 0$ as

$$V(s) = \frac{1}{2\left(\frac{ds}{dr}\right)^2}\left(\frac{dp}{dr} - \frac{3}{2\left(\frac{ds}{dr}\right)^2}\left(\frac{d^2s}{dr^2}\right)^2 + \frac{1}{\left(\frac{ds}{dr}\right)}\frac{d^3s}{dr^3} + \frac{p^2}{2} - 2q\right). \tag{C9}$$

□

Remark. Applying this to $s = r_*(r) = r + 2M\log|r - 2M|$ *gives*

$$V(r(r_*)) = \frac{(r-2M)^2}{2r^2}\left(\frac{df}{dr} + \frac{2M(2r-3M)}{r^2(r-2M)^2} + \frac{p^2}{2} - 2q\right). \tag{C10}$$

APPENDIX D: USEFUL RESULTS FROM ANALYSIS

1. Sobolev embedding

Theorem D.1 (Local Compactness of the H^s Sobolev Injection). *Let* $d \geq 1$, $s > 0$, *and*

$$p_c = \begin{cases} \dfrac{2d}{d-2s} & s < \dfrac{d}{2} \\ \infty & \textit{otherwise}. \end{cases} \tag{D1}$$

Then, the embedding $H^s(\mathbb{R}^d) \hookrightarrow L^p_{\text{loc}}(\mathbb{R}^d)$ *is compact* $\forall 1 \leq p < p_c$. *In other words, for* $(f_n)_n \subset H^s(\mathbb{R}^d)$ *bounded, there exists* $f \in H^s(\mathbb{R}^d)$ *and a subsequence* $(f_{n_m})_m$ *such that*

$$f_{n_m} \rightharpoonup f \qquad H^s(\mathbb{R}^d), \tag{D2}$$

$$f_{n_m} \to f \qquad L^p_{\text{loc}}(\mathbb{R}^d) \quad \forall 1 \leq p < p_c. \tag{D3}$$

Proof. This result can be found in any text on Sobolev spaces, for example, Ref. 32.

□

2. The multiplication operator is compact

Proposition D.2. Let $q \in C^0(\mathbb{R}^n, \mathbb{R})$ *with* $\lim_{|x|\to\infty} q(x) = 0$ *and* $s > 0$. *Then,* $M_q : u \to qu$ *is a compact operator from* $H^s(\mathbb{R}^n, \mathbb{R})$ *to* $L^2(\mathbb{R}^n, \mathbb{R})$.

Proof. The function q is continuous and decays; hence, it is bounded. Let $\epsilon > 0$, then by assumption, $\exists R > 0$ such that

$$|q(x)| \leq \epsilon \qquad \text{if } |x| \geq R. \tag{D4}$$

Define $\chi_R : \mathbb{R} \to \mathbb{R}$ smooth by

$$\chi_R(x) = \begin{cases} 1, & |x| \leq R \\ 0, & |x| \geq R + 1. \end{cases} \tag{D5}$$

Let $(f_n)_n \subset H^s(\mathbb{R}^n, \mathbb{R})$ be bounded, so local compactness of the Sobolev embedding (Theorem D.1) gives convergence in $H^s(\mathbb{R}^n, \mathbb{R})$ and weak convergence in $L^2_{\text{loc}}(\mathbb{R}^n, \mathbb{R})$ up to a subsequence. Let the limit be $f \in H^s(\mathbb{R}^n, \mathbb{R})$. Therefore,

$$\|\chi_R q f_{m_n} - \chi_R q f\|^2_{L^2(\mathbb{R}^n)} = \|\chi_R q f_{m_n} - \chi_R q f\|^2_{L^2(B_{R+1}(0))}, \tag{D6}$$

$$\leq C \sup_{x\in\mathbb{R}} |q(x)|^2 \|f_{m_n} - f\|^2_{L^2(B_{R+1}(0))} \leq \epsilon^2. \tag{D7}$$

Furthermore, consider the set $S_R := \{\chi_R q f : f \in H^s(\mathbb{R}^n, \mathbb{R}), \|f\|_{H^s(\mathbb{R}^n)} \leq 1\}$. Then,

$$\|(1-\chi_R) q f\|_{L^2(\mathbb{R}^n)} \leq \epsilon^2 \|f\|_{L^2(\mathbb{R}^n)} \leq \epsilon^2. \tag{D8}$$

Hence, S_∞ is within a ϵ-neighborhood of S_R, which is compact; therefore, S_∞ is compact. By the characterization of compactness through weak convergence, $q\,\mathrm{f}_m \to qf$ in $L^2(\mathbb{R}^n, \mathbb{R})$ up to a subsequence.

□

3. A regularity result

Theorem D.3 (Regularity for the Schrödinger Equation). *Let $u \in H^1(\mathbb{R})$ be a weak solution of the equation $(-\Delta + V)u = \lambda u$, where V is a measurable function and $\lambda \in \mathbb{C}$. Then, if $V \in C^\infty(\Omega)$ with $\Omega \subset \mathbb{R}$ open, not necessarily bounded, then $u \in C^\infty(\Omega)$ also.*

Proof (Ref. 33, Vol. II, p. 55). Note one can argue this from standard elliptic regularity results and Sobolev embeddings. In this paper, only the one-dimensional case of this is applied, which is completely elementary.

□

APPENDIX E: A RESULT ON STABILITY IN SPHERICAL GAUGE

This section contains a few technical results on where the instability may lie in frequency space. This helped guide the search for a suitable test function and the subsequent instability.

Proposition E.1. Consider the quartic polynomial

$$P(x) = ax^4 + bx^3 + cx^2 + dx + e. \tag{E1}$$

Let Δ denote its discriminant and define

$$\Delta_0 = 64a^3 e - 16a^2c^2 + 16ab^2c - 16a^2bd - 3b^4. \tag{E2}$$

If $\Delta < 0$, then $P(x)$ has two distinct real roots and two complex conjugate roots with non-zero imaginary part. If $\Delta > 0$ and $\Delta_0 > 0$, then there are two pairs of complex conjugate roots with non-zero imaginary part.

Proof. See Ref. 34.

□

Proposition E.2 (regions of stability in frequency space). Let $\mu > 0$ and $k \neq 0$. There does not exist a solution $\mathfrak{h}$ of ODE (3.5) with $c_1 = 0$, $k_2 = 0$, and $\hat{k} \in \mathbb{R}\backslash(-\sqrt{2}, \sqrt{2})$ or *$\hat{\mu} \geq \frac{3}{8}\sqrt{\frac{3}{2}}$.*

Proof. From Proposition 3.8, the admissible boundary conditions for the solution are $\mathfrak{h}(r) = k_1 \mathfrak{h}^{2M,+}(r)$ at the future event horizon and $\mathfrak{h}(r) = c_2 \mathfrak{h}_z^{\infty,-}(r)$ at spacelike infinity. Without loss of generality, take $k_1 > 0$. Now, since the solution must decay exponentially towards infinity, there must be maxima $a \in (1, \infty)$. At such a point, one has

$$\frac{d^2\mathfrak{h}}{dr^2}(a) = \frac{a(\hat{\mu}^2 a + \hat{k}^2(\hat{\mu}^2 a^4 - 2a + 2) + \hat{k}^4 a^3(a-1))}{(\hat{k}^2 a^3 + 1)(a-1)^2} \mathfrak{h}(a), \tag{E3}$$

with $\mathfrak{h}(a) > 0$. To derive a contradiction, one must have

$$\frac{a(\hat{\mu}^2 a + \hat{k}^2(\hat{\mu}^2 a^4 - 2a + 2) + \hat{k}^4 a^3(a-1))}{(\hat{k}^2 a^3 + 1)(a-1)^2} > 0. \tag{E4}$$

A sufficient condition for the numerator to be positive is

$$\hat{\mu}^2 a^4 - 2a + 2 \geq 0. \tag{E5}$$

This has the discriminant

$$\Delta = 16\hat{\mu}^4(128\hat{\mu}^2 - 27), \qquad \Delta_0 = 128\hat{\mu}^2. \tag{E6}$$

Hence, if $\hat{\mu}^2 > \frac{27}{128}$, then there are no real roots. Thus, because the polynomial is positive at a point, say $a = 1$, it is positive everywhere. If $\Delta = 0$, there is a double real root and two complex conjugate roots. The real roots can only occur at a stationary point of the polynomial, and therefore, the polynomial cannot be negative anywhere. Since all other terms in the numerator are positive, the prefactor of $\mathfrak{h}$ also is. Hence, there can be no solution with the conditions $k_2 = 0$ and $c_1 = 0$ if $\hat{\mu} \geq \frac{3}{8}\sqrt{\frac{3}{2}}$.

Another sufficient condition for positivity of the numerator is

$$\hat{k}^2 a^3 - 2 \geq 0. \tag{E7}$$

This polynomial has a single real root at $a = \left(\frac{2}{\hat{k}^2}\right)^{\frac{1}{3}}$. For positivity on $a \in (1, \infty)$, one requires $\frac{2}{\hat{k}^2} \leq 1$ or $\hat{k}^2 \geq 2$. Note that if $\hat{\mu} = 0$, then this is precisely the polynomial that governs positivity. Hence, this bound for $\hat{k}$ is sharp.

□

Remark. By an almost identical argument, one can make the bound for $\hat{\mu}$ even sharper and show that $\hat{\mu} < \frac{1}{4}$ and $\hat{\mu} \leq \sqrt{2}|\hat{k}|$.

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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